The Algebraic Theory of Integration

Draft Lecture Notes

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Introduction

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1. Integration of Rational Functions

It is well-known that the integral of a rational function

$$\int \frac{p(x)}{q(x)}\,dx\,,\quad p(x)\,,q(x)\in\mathbb{Q}[x]\,,\quad \mathrm{GCD}\,(p,q)=1\,,\quad q \text{ monic}\,,$$

can be expressed as

$$\frac{g(x)}{q(x)}+c_1\log(x-\alpha_1)+\ldots+c_n\log(x-\alpha_n)$$

where $g(x)\in\mathbb{Q}[x]\,,\ \alpha_1,\ldots,\alpha_n$ the distinct roots of q and $c_1,\ldots,c_n\in\mathbb{Q}(\alpha_1,\ldots,\alpha_n)$. Standard calculus textbooks usually suggest the use of a partial fraction decomposition and the Hermite trick to obtain g. The algorithm goes the following way:

Step 1: For the squarefree decomposition of

$$q=f_1f_2^2\cdots f_r^r\,,\quad f_i\in\mathbb{Q}[x]\text{ squarefree}\,,\quad \mathrm{GCD}\,(f_i,f_j)=1\,,\quad\text{for}\quad 1\le i<j\le r$$

compute the partial fraction decomposition of p/q,

$$\frac{p}{q}=g_0+\frac{g_{11}}{f_1}+\frac{g_{21}}{f_2}+\frac{g_{22}}{f_2^2}+\cdots+\frac{g_{r1}}{f_r}+\cdots+\frac{g_{rr}}{f_r^r}\,,$$
$$\text{where}\quad g_0,g_{ij}\in\mathbb{Q}[x]\,,\qquad \deg\,(g_{ij})<\deg\,(f_i)\quad\text{for}\quad 1\le j\le i\le r\,.$$

$\int g_0$ is easy to find so we shall assume from now on that $\deg\,(p)<\deg\,(q)$.

Step 2: From highest to lowest power, reduce each integral of the form

$$\int\frac{g(x)}{f^n(x)}\,dx\,,\quad \deg\,(g)<\deg\,(f)\,,\quad f\text{ squarefree}\,,\quad n\ge 2\,,$$

to an integral

$$\int\frac{h(x)}{f^{n-1}(x)}\,dx\,,\quad \deg\,(h)<\deg\,(f)\,.$$

This we do in the following way: Since f is squarefree, $\mathrm{GCD}\,(f(x), f'(x)) = 1$, $f'(x)$ denoting the derivative df/dx. Therefore we can compute polynomials $u, v \in \mathbb{Q}[x]$ such that

$$g(x) = u(x)f(x) + v(x)(1-n)f'(x)\,, \qquad \deg(u) < \deg(f)\,,\ \deg(v) < \deg(f)\,.$$

Hence

$$\int \frac{g(x)}{f^n(x)}\,dx = \int \frac{u(x)}{f^{n-1}(x)}\,dx + \int v(x)\frac{(1-n)f'(x)}{f^n(x)}\,dx$$

the second integral of which can be reduced by parts to

$$\frac{v}{f^{n-1}} - \int \frac{v'}{f^{n-1}}\,, \quad \text{thus} \quad \int \frac{g}{f^n} = \frac{v}{f^{n-1}} + \int \frac{u-v'}{f^{n-1}}$$

and we get $h = u - v'$ satisfying the asserted degree condition. After eliminating all recursively appearing

$$\int \frac{g_{ij}}{f_i^j}\,, \quad 1 \le i \le r\,, \quad 2 \le j \le i\,,$$

we can again bring both the remaining integrals

$$\int \frac{h_i(x)}{f_i(x)}\,dx \quad \text{and the already known rational parts} \quad \frac{v_i}{f_i^j}\,,\ 1 \le j < i\,,$$

to a common denominator. This proves the existence of two polynomials $g(x), h(x)$ such that

$$\int \frac{p(x)}{q(x)}\,dx = \frac{g(x)}{f_2(x)\cdots f_r^{r-1}(x)} + \int \frac{h(x)}{f_1(x)\cdots f_r(x)}\,dx \tag{1.1}$$

and $\quad \deg(g) < \deg(f_2) + \ldots + (r-1)\deg(f_r)\,, \quad \deg(h) < \deg(f_1) + \ldots + \deg(f_r)\,.$

Another algorithm for computing g and h is to start with unknown coefficients for these polynomials, differentiate (1.1), remove the denominators and equate the coefficients of equal powers of x^i. In fact, (1.1) then can be written as

$$p = \left(\prod_{i=1}^{r} f_i\right) g' - \left(f_1 \sum_{i=2}^{r} (i-1) f_i' \prod_{\substack{j=2 \\ j \ne i}}^{r} f_j\right) g + \left(\prod_{i=2}^{r} f_i^{i-1}\right) h \tag{1.2}$$

which leads to a linear system in $\deg(q)$ equations and $\deg(q)$ unknowns. If we set

$$q^* = \prod_{i=1}^{r} f_i\,, \quad \bar{q} = \prod_{i=2}^{r} f_i^{i-1}$$

then $\mathrm{GCD}\,(q, q') = \bar{q}$ and $q/\bar{q} = q^*$. Furthermore, the coefficient of g in (1.2) is $q^*\bar{q}'/\bar{q} = q'/\bar{q} - q^{*\prime}$ and we therefore can set up the system without completing the squarefree decomposition of q. We now show that the polynomials g and h are uniquely determined. For this we shall prove a very elementary lemma, but which will also have important ramifications later on.

Lemma 1.1: Let $u, v \in \mathbb{Q}[x]$, $\mathrm{GCD}\,(u,v) = 1$ and let $(u/v)' = p/q$, $p, q \in \mathbb{Q}[x]$, $\mathrm{GCD}\,(p,q) = 1$. Assume that $w \in \mathbb{Q}[x]$ is squarefree such that w divides q. Then w divides v and if r is the multiplicity of w in v, w^{r+1} must divide q.

Proof: First it is easy to realize that we can restrict ourselves to w being irreducible. Otherwise, since w is squarefree, applying the lemma for all irreducible factors of w leads to the one for w itself. Notice that r is the minimal multiplicity with which any irreducible factor of w occurs in q. Now, since

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{p}{q}$$

it is clear that w must divide v. Assume now that $v = w^r \hat{w}$ with $\mathrm{GCD}\,(w, \hat{w}) = 1$. We show that w^r does not divide $u'v - uv'$. Suppose the contrary. Since w^r divides $u'v$ and $\mathrm{GCD}\,(w,u) = 1$, w^r then would have to divide $v' = rw^{r-1}w'\hat{w} + w^r\hat{w}'$, hence w needed to divide $w'\hat{w}$. But this is impossible since w is squarefree. Therefore w^{r+1} must remain in the reduced denominator of $(u/v)'$. $\diamondsuit$

It follows immediately that the solution to (1.1) is unique. For by lemma 1.1, the only solution with $p = 0$ is $g = h = 0$ because $(g/\bar{q})' = -h/q^*$ and q^* is squarefree. We have incidentally also shown that $\int h/q^*$ cannot be a rational function. Of course, this was to be expected since $\int h/q^*$ can be computed in the following well-known way: Let $q^*(x) = (x - \alpha_1) \cdots (x - \alpha_k)$, where $\alpha_1, \ldots, \alpha_k$ are the roots of q^*. Since q^* is squarefree, these roots are distinct. Then

$$\int \frac{h(x)}{q^*(x)}\,dx = \sum_{i=1}^{k} c_i \log\,(x - \alpha_i) \quad \text{with} \quad c_i = \frac{h(\alpha_i)}{q^{*\prime}(\alpha_i)}\,, \quad 1 \le i \le k\,.$$

To obtain the identity for c_i just set $x = \alpha_i$ in $h = \sum_{i=1}^{k} c_i \prod_{j \ne i}(x - \alpha_j)$.

Example 1.1: $\displaystyle\int \frac{8x}{x^4 - 2}\,dx.$

Since $(x^4 - 2) = (x - \sqrt[4]{2})(x + \sqrt[4]{2})(x - i\sqrt[4]{2})(x + i\sqrt[4]{2})$ and

$$c_i = \left.\frac{8x}{(x^4 - 2)'}\right|_{\alpha_i} = \frac{2}{\alpha_i^2}\,, \quad c_1 = c_2 = \sqrt{2}\,, \quad c_3 = c_4 = -\sqrt{2}\,,$$

$$\begin{aligned}\int \frac{8x}{x^4 - 2}\,dx &= \sqrt{2}\,\left(\log\,(x - \sqrt[4]{2}) + \log\,(x + \sqrt[4]{2})\right) - \sqrt{2}\,\left(\log\,(x + i\sqrt[4]{2}) + \log\,(x - i\sqrt[4]{2})\right) \\ &= \sqrt{2}\,\log\left(x^2 - \sqrt{2}\right) - \sqrt{2}\,\log\left(x^2 + \sqrt{2}\right)\,.\end{aligned}$$

This example illustrates that not the entire splitting field of q^* may be necessary to express the integral. The question arises how to determine the minimal extension of $\mathbb{Q}$ in which we can express a given rational integral. Before we formulate the solution to this problem we describe a canonical representation for the answer to the integral.

Definition 1.1: The expression

$$c_1 \frac{v_1'}{v_1} + \ldots + c_n \frac{v_n'}{v_n}\,, \quad c_i \in \bar{\mathbb{Q}}\,,\ v_i \in \bar{\mathbb{Q}}[x]\,,$$

$\bar{\mathbb{Q}}$ denoting the algebraic closure of $\mathbb{Q}$, is in *canonical form* if v_i non-constant, monic and squarefree, $1 \le i \le n$, and $c_i \ne c_j$ and $\mathrm{GCD}\,(v_i, v_j) = 1$ for all $1 \le i < j \le n$.

Obviously, for every rational function p/q, $\deg(p) < \deg(q)$, q squarefree, there exists a canonical form equal to it. In fact, a canonical simplifier for expressions of the form $\sum_{i=1}^{n} c_i v_i'/v_i$ proceeds as follows.

Step 1: If v_i and v_j are not relatively prime to one another, replace

$$c_i \frac{v_i'}{v_i} + c_j \frac{v_j'}{v_j} \quad \text{by} \quad c_i \frac{(v_i/g)'}{v_i/g} + c_j \frac{(v_j/g)'}{v_j/g} + (c_i + c_j) \frac{g'}{g}$$

where $g = \mathrm{GCD}\,(v_i, v_j)$. Since we steadily decrease the degrees of the v_i's this process will yield a new expression $\sum_{i=1}^{\bar{n}} \bar{c}_i \bar{v}_i'/\bar{v}_i$, where the $\bar{v}_i$ are relatively prime to one another.

Step 2: If, at this point, any of the $\bar{v}_i$ is not monic, replace

$$\bar{c}_i \, \frac{\bar{v}_i'}{\bar{v}_i} \quad \text{by} \quad \bar{c}_i \, \frac{(\bar{v}_i/\mathrm{ldcf}\,(\bar{v}_i))'}{\bar{v}_i/\mathrm{ldcf}\,(\bar{v}_i)} \,.$$

Then squarefree decompose each $\bar{v}_i$ into $\bar{v}_{i_1} \ldots \bar{v}_{i_{r_i}}^{r_i}$ and replace

$$\bar{c}_i \frac{\bar{v}_i'}{\bar{v}_i} \quad \text{by} \quad \bar{c}_i \frac{\bar{v}_{i_1}'}{\bar{v}_{i_1}} + 2\bar{c}_i \frac{\bar{v}_{i_2}'}{\bar{v}_{i_2}} + \ldots + r_i \bar{c}_i \frac{\bar{v}_{i_{r_i}}'}{\bar{v}_{i_{r_i}}}$$

This step yields an expression $\sum_{i=1}^{\hat{n}} \hat{c}_i \hat{v}_i'/\hat{v}_i$ such that the $\hat{v}_i$ are squarefree and relatively prime to one another.

Step 3: Finally, sum up terms with equal constants, i.e. if $\hat{c}_i = \hat{c}_j$ then replace

$$\hat{c}_i \frac{\hat{v}_i'}{\hat{v}_i} + \hat{c}_j \frac{\hat{v}_j'}{\hat{v}_j} \quad \text{by} \quad \hat{c}_i \frac{(\hat{v}_i \hat{v}_j)'}{\hat{v}_i \hat{v}_j} \,.$$

Integrating each term in our replacement rules and applying the rules for logarithms easily proves our replacements correct, though we could also simply apply the product rule for differentiation. We also notice that the canonical expression will not require constants or polynomial coefficients from outside the field in which the original expression is given. Therefore, it suffices to describe the field in which a canonical form equivalent of the rational function p/q lies.

Theorem 1.1: Let $p, q \in \mathbb{Q}[x]$, $\deg(p) < \deg(q)$, q squarefree, $\mathrm{GCD}\,(p, q) = 1$. Then $c_1 v_1'/v_1 + \ldots + c_n v_n'/v_n$ is a canonical form equal to p/q if and only if $c_1, \ldots, c_n$ are all distinct roots of $R(c) = \mathrm{res}_x(q(x), p(x) - c q'(x)) \in \mathbb{Q}[c]$ and $v_i = \mathrm{GCD}\,(q, p - c_i q')$, made monic.

Proof: Only if: Assume $\sum_{i=1}^{n} c_i v_i'/v_i = p/q$ is in canonical form. We shall prove the following statements.

(1) $p = \sum_{i=1}^{n} (c_i v_i' \prod_{\substack{j=1 \\ j \ne i}}^{n} v_j)$ and $q = \prod_{i=1}^{n} v_i$ provided we assume, as we may, that q is monic.

(2) $\mathrm{GCD}\,(p - c_i q', v_j) = v_i$, if $i = j$, and 1 if $i \neq j$.

(3) For any root d of $R(c)$ there exists an index i_0 such that $d = c_{i_0}$.

The statements (1) – (3) obviously establish this part of the theorem since (1) and (2) imply that

$$\mathrm{GCD}\,(p - c_i q', q) = v_i\,, \quad 1 \leq i \leq n\,,$$

which in turn shows that $R(c_i) = 0$. (See also the remark following the proof.) Furthermore, (3) proves that every root of R actually occurs as a constant in our canonical form. As said before, we may assume without loss of generality that q is monic since both

$$\mathrm{res}_x \left(\frac{q}{\mathrm{ldcf}\,(q)}\,,\ \frac{p}{\mathrm{ldcf}\,(q)} - c\,\frac{q'}{\mathrm{ldcf}\,(q)} \right)$$

and

$$\mathrm{GCD} \left(\frac{q}{\mathrm{ldcf}\,(q)}\,,\ \frac{p}{\mathrm{ldcf}\,(q)} - c_i\,\frac{q'}{\mathrm{ldcf}\,(q)} \right)$$

are scalar multiples of $R(c)$ and $\mathrm{GCD}\,(q, p - c_i q')$. Let $v_i^* = \prod_{\substack{j=1 \\ j \neq i}}^{n} v_j$. Then the canonical form transforms into

$$p \prod_{i=1}^{n} v_i = q \sum_{i=1}^{n} c_i v_i' v_i^*\,.$$

Since $\mathrm{GCD}\,(p, q) = 1\,,\ q \mid \prod_{i=1}^{n} v_i$. On the other hand,

$$\mathrm{GCD}\,(v_j, \sum_{i=1}^{n} c_i v_i' v_i^*) = \mathrm{GCD}\,(v_j, c_j v_j' v_j^*) = 1$$

because v_j is squarefree, thus $v_j \mid q$. The monicity assumptions for v_j and q show that $q = \prod_{i=1}^{n} v_i$. This proves (1).

The next statement follows similarly. For

$$p - c_i q' = \sum_{j=1}^{n} (c_j - c_i) v_j' v_j^*$$

and thus $v_i \mid p - c_i q'$ and $\mathrm{GCD}\,(p - c_i q', v_j) = \mathrm{GCD}\,((c_j - c_i) v_j' v_j^*\,,\ v_j) = 1$, for $i \neq j$, since $c_i \neq c_j$.

Finally, let $R(d) = 0$ which implies that $w = \mathrm{GCD}\,(p - dq', q)$ is not constant. (See also the remark following the proof.) Let α be a root of w. Since $w \mid q\,,\ \alpha$ must be a root of exactly one v_{i_0}. But $0 = (p - dq')\alpha = c_{i_0} - d v_{i_0}' \alpha v_{i_0}^* \alpha$ and $v_{i_0}' \alpha \neq 0\,,\ v_{i_0}^* \alpha \neq 0$, hence $c_{i_0} - d = 0$.

If: From our canonical simplifier and the remarks before we know the existence of some canonical form $\sum_{i=1}^{m} d_i v_i' / v_i = p/q$. Applying the *Only If* part to this form proves the statement. $\diamondsuit$

Remark: In the proof of this theorem we have twice used the fact that for $f(x) = a_l x^l + \ldots + a_0$, $g(x) = b_m x^m + \ldots + b_0 \in D[x]$, D an integral domain, $a_l b_m \neq 0$, the resultant of f and g, $\mathrm{res}\,(f, g)$,

$$\det \begin{vmatrix} a_l & a_{l-1} & \ldots & & & & a_1 & a_0 & & & \\ & a_l & a_{l-1} & \ldots & & & a_2 & a_1 & a_0 & & \\ & & \ddots & & & & & & & \ddots & \\ & & & a_l & a_{l-1} & \ldots & a_m & \ldots & & a_1 & a_0 \\ b_m & b_{m-1} & \ldots & b_1 & b_0 & & & & & & \\ & b_m & b_{m-1} & \ldots & b_1 & b_0 & & & & & \\ & & \cdot & & & & \cdot & & & & \\ & & & \cdot & & & \cdot & \cdot & & & \\ & & & & \cdot & & \cdot & & \cdot & & \\ & & & & & \cdot & \cdot & & & \cdot & \\ & & & & & & b_m & b_{m-1} & \ldots & b_1 & b_0 \end{vmatrix}$$

is zero if and only if $\mathrm{GCD}\,(f, g)$ is non-constant in $\mathrm{QF}(D)[x]$. Closer inspection shows, however, that we applied this fact to homomorphic images of f and g. If $\phi\colon D \to D$ is a homomorphism (in our case evaluation of polynomials in c at points c_i), then in general

$$\mathrm{res}\,(\phi f, \phi g) \neq \phi(\mathrm{res}\,(f, g))\,,$$

where $\phi f = (\phi a_l)x^l + \ldots + (\phi a_0)$, $\phi g = (\phi b_m)x^m + \ldots + (\phi b_0)$. However, if $a_l = 1$, then $\mathrm{res}\,(\phi f, \phi g) = \phi(\mathrm{res}\,(f, g))$ as can be seen from the above determinant definition. (By definition, we assume that $\mathrm{res}\,(0, g) = 0$.) This is true for our application since q is monic.

Example 1.1 (continued): Applying theorem 1.1 we get

$$\begin{aligned} R(c) &= \mathrm{res}\,(8x - c(x^4-2)', x^4 - 2) \\ &= \det \begin{pmatrix} -4c & 0 & 8 & 0 & & & \\ & -4c & 0 & 8 & 0 & & \\ & & -4c & 0 & 8 & 0 & \\ & & & -4c & 0 & 8 & 0 \\ 1 & 0 & 0 & 0 & -2 & & \\ & 1 & 0 & 0 & 0 & -2 & \\ & & 1 & 0 & 0 & 0 & -2 \end{pmatrix} = -2048(c^2-2)^2 \end{aligned}$$

Hence $c_1 = -\sqrt{2}$ and $c_2 = \sqrt{2}$. The Euclidean algorithm in $\mathbb{Q}(\sqrt{2})[x]$ computes the $\mathrm{GCD}\,(4\sqrt{2}x^3 + 8x, x^4 - 2)$ as follows.

$$\begin{array}{r|lll} & \frac{1}{4\sqrt{2}}x & & \\ \hline 4\sqrt{2}x^3 + 8x & x^4 & & -2 \\ & x^4 & +\sqrt{2}x^2 & \\ \hline & & -\sqrt{2}x^2 & -2 \end{array} \qquad \begin{array}{r|l} & -4x \\ \hline -\sqrt{2}x^2 - 2 & 4\sqrt{2}x^3 + 8x \\ & 4\sqrt{2}x^3 + 8x \\ \hline & 0 \end{array}$$

hence $v_1 = -\frac{1}{\sqrt{2}}(-\sqrt{2}x^2 - 2) = x^2 + \sqrt{2}$ and $v_2 = x^2 - \sqrt{2}$.

One can observe that the second term in the sum is like the first with $-\sqrt{2}$ replaced by $\sqrt{2}$. It follows from theorem 1.1 that for each constant c_{i_1} its conjugates $c_{i_2}, \ldots, c_{i_k}$ also appear in the sum. Notice that $(c - c_{i_1})(c - c_{i_2}) \cdots (c - c_{i_k}) \in \mathbb{Q}[c]$ is an irreducible factor of $R(c)$. The corresponding v_{i_j} are obtained by applying the automorphism $c_{i_1} \mapsto c_{i_j}$ to the coefficients of v_{i_1}.

In summary, the integral $\int p/q$ can be expressed as a rational function and a sum of logarithms, which we call the *rational* and *logarithmic* parts, resp. By Lemma 1.1 we have already established that the logarithmic part is not a rational function. At this point, we are not quite in the position of showing that this part is transcendental and shall insert its proof after theorem 2.1.

One final remark on the algebraic representation of the necessary field extension is in order. There are at least two possibilities:

1) For each c_i we give its minimal polynomial and a pair (z, ϵ), $z \in \mathbb{Q}(\sqrt{-1})$, $\epsilon \in \mathbb{Q}$, such that the box

$$B = \{\xi \mid Re(z) \leq Re(\xi) \leq Re(z) + \epsilon\,,\ Im(z) \leq Im(\xi) \leq Im(z) + \epsilon\}$$

   contains c_i and none of its conjugates. This representation is very useful when numeric evaluations of the integral are anticipated. We wish to emphasize that z and ϵ can be efficiently computed due to the results by A. Schönhage [83].

2) We construct a primitive element γ for the splitting field of $R(c)$ in terms of its minimal polynomial $m(\gamma)$ and represent each $c_i \in \mathbb{Q}[\gamma]/(m(\gamma))$. This representation has the disadvantage that $\deg(m)$ may be as large as $\deg(q)!$, but on the other hand the arithmetic with the c_i may be more transparent.

Historic Remarks: The basic algorithm seems to be known at least since C. Hermite, published posthumously [1912]. The idea of solving (1.1) directly as linear system in the unknown coefficients comes from E. Horowitz [71]. R. Tobey [67] recognized the phenomenon that a smaller field than the splitting field of q can be sufficient to express the integral. Theorem 1.1 was discovered by M. Rothstein [76] and, in its essence, also by B. Trager [76]. In section 4 we will present another proof for a generalization of this theorem. The asymptotic complexity of Hermite's algorithm was studied in detail by D. Yun [77].

2. Differential Fields and Liouville Extensions

Definition 2.1: Let R be a commutative ring with unity. A *derivation* $'$ is a map from R into R, $r \mapsto r'$, such that for all $r, s \in R$

$$(r+s)' = r' + s', \quad (rs)' = r's + rs'.$$

R together with its derivation is called a *differential ring*.

Since $1' = (1^2)' = 2 \cdot 1 \cdot 1'$, $1' = 0$, and thus the set $C_R = \{c \mid c \in R,\ c' = 0\}$ forms a subring of R, the *ring of constants* in R.

Let $r \in R$ possess a multiplicative inverse r^{-1}. Then $0 = (rr^{-1})' = r'r^{-1} + r(r^{-1})'$, hence $(r^{-1})' = -r'r^{-2}$. Therefore, $(r^n)' = nr^{n-1}r'$ for such r and all $n \in \mathbb{Z}$.

Lemma 2.1: Let the differential ring R be an integral domain. Then the derivation $'$ extends uniquely to the quotient field of R, $\mathrm{QF}(R)$.

Proof: The embedding of R into $\mathrm{QF}(R)$ requires us to define $[r/1]' = [r'/1]$ where $r \in R$ and $[r/s]$ denotes the equivalence class for the pair r/s. Since $[1/r]$ is the multiplicative inverse of $[r/1]$, $[1/r]' = -[r'/1][1/r^2]$, $r \neq 0$. Hence we must define

$$[r/s]' = [(r's - rs')/s^2], \quad r, s \in R,\ s \neq 0.$$

It is left to the reader to show that this definition is independent of the choice of representative of $[r/s]$ and in fact constitutes a derivation on $\mathrm{QF}(R)$. $\diamondsuit$

The proof also shows that $C_{\mathrm{QF}(R)}$ is in fact a subfield of $\mathrm{QF}(R)$. In case R itself is already a field, we shall call it a *differential field* and C_R its *field of constants*.

Example 2.1: Consider the field M of meromorphic functions on the complex plane, i.e. functions which are analytic everywhere except at possibly infinitely many isolated singularities which must be poles. C_M is obviously $\mathbb{C}$, but we shall be interested in differential subfields of M with possibly smaller constant fields.

1. $\mathbb{Q}$: This is the smallest subfield of M but it shall be noted that the only derivation on $\mathbb{Q}$ is the one for which $C_{\mathbb{Q}} = \mathbb{Q}$
2. $\mathbb{Q}(x)$: Though this field is a differential subfield of M, indefinite integration may give anti-derivatives outside this field, as proven in section 1. This field is, however, closed under function composition.
3. $\mathbb{Q}(x, \exp(x))$: We will prove after theorem 2.3 that $\exp(x)$ is a transcendental function over $\mathbb{Q}(x)$. Notice that this field also contains $\cosh(x) = (\exp(x) + 1/\exp(x))/2$. Using the exp function, similar fields can be constructed containing the trigonometric functions. Again anti-derivatives may lie outside the field, but something more fundamentally different can happen: E.g., $\int \exp(x)/x\,dx$ cannot be written even as a "closed form expression," a notion which we will make precise in this section. It is important to realize that this field does not remain closed under function composition, in fact $\exp(1/x)$ has an essential singularity at 0. Of course, $\exp(1/x)$ remains meromorphic on $\mathbb{C} \setminus \{0\}$.

Therefore we shall allow functions in our subfields which are meromorphic on only an open subregion of $\mathbb{C}$.

4. $\mathbb{Q}(x, \log(x))$: Since the logarithm is a multivalued function such as $\sqrt{x}$, we must again restrict our region of definition, now to $\mathbb{C} \setminus \{x + \mathbf{i} \cdot 0 \mid x \geq 0\}$, and we shall furthermore agree that
$$\log(x) = \log|x| + \mathbf{i} \cdot \arg(x)\,,\ 0 < \arg(x) < 2\pi\,.$$
We will prove in a remark after theorem 2.1 that $\log(x)$ is a transcendental function over $\mathbb{Q}(x)$. Again, anti-derivatives in closed form may not exist, an example being $\int dx/\log(x)$. Finally, some inverse trigonometric functions can be found in similar fields, e.g.
$$\arctan(x) \in \mathbb{Q}\left(\mathbf{i},\, x,\, \log\frac{\mathbf{i}+x}{\mathbf{i}-x}\right)\,.$$

The previous example did not contain any algebraic function fields. For this case we will develop the abstract algebraic theory in detail now.

Theorem 2.1: Let K be a differential field and $K(\theta)$ a separable algebraic extension of K (i.e. the minimal polynomial $h(\theta) \in K[\theta]$ of θ has no repeated roots.) Then the derivation $'$ of K extends uniquely to a derivation on $K(\theta)$.

Proof: Define $D_0\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^n a_i' x^i$ and $D_1\left(\sum_{i=0}^n a_i x^i\right) = \sum_{i=0}^n i a_i x^{i-1}$ to be operators on $K[x]$. Then if $K(\theta)$ has a derivation $'$ extending that of K we must have
$$0 = h(\theta)' = (D_0 h)(\theta) + (D_1 h)(\theta)\theta'\,.$$
Since θ is separable, $D_1 h \neq 0$, and since h is the minimal polynomial for θ, $(D_1 h)(\theta) \neq 0$. Thus $\theta' = -(D_0 h)(\theta)/(D_1 h)(\theta)$. Now every element of $K(\theta)$ can be expressed as a polynomial in $K[\theta]$, therefore the extended derivation is unique.

The hard part is to show that this map is well-defined and constitutes a derivation. To prove this, we shall redefine $'$: Let D be an operator on $K[x]$ defined as
$$Df = D_0 f + g(x) D_1 f\,, \quad f \in K[x]\,,$$
where g is to be determined later. Then $D(f_1 + f_2) = Df_1 + Df_2$ and $D(f_1 f_2) = (Df_1)f_2 + f_1(Df_2)$ since the analogous identities hold for both D_0 and D_1. Note also that $Da = a'$ for any $a \in K$. We now prove that the epimorphism
$$\phi\colon K[x] \to K(\theta)\,, \quad f(x) \mapsto f(\theta)$$
defines a derivation $(\phi f)' = \phi(Df)$ in $K(\theta)$. Since D is a derivation on $K[x]$, $'$ is one on $K(\theta)$, provided it is well-defined. Therefore we must show that $\phi f_1 = \phi f_2$ implies $(\phi f_1)' = (\phi f_2)'$. The first identity is equivalent to $h \mid f_1 - f_2$, the second to $h \mid D(f_1 - f_2)$. Let $f_1 - f_2 = h\bar{h}$. Then $h \mid D(h\bar{h})$ if $h \mid D(h)$. But this condition is equivalent to $h \mid D_0 h + g(x) D_1 h$ meaning $(D_0 h)(\theta) + g(\theta)(D_1 h)(\theta) = 0$. Since $(D_1 h)(\theta) \neq 0$, a polynomial g satisfying this condition can be found. $\diamondsuit$

Corollary: Let $N = K(\theta)$ be a normal separable algebraic extension of the differential field K and let σ be an element of the automorphism group on N fixing K. Then

$$\sigma(a') = (\sigma a)' \quad \text{for all } a \in N\,.$$

Proof: Let $\theta_2, \ldots, \theta_n$ be the conjugates of $\theta = \theta_1$. There exists an index i, $1 \le i \le n$ such that $\sigma(\theta) = \theta_i$. From the previous theorem we know for $a = f(\theta)$ that

$$a' = (Df)(\theta) = (D_0 f)(\theta) + g(\theta)(D_1 f)(\theta)$$

hence $\sigma(a') = (D_0 f)(\theta_i) + g(\theta_i)(D_1 f)(\theta_i) = (Df)(\theta_i) = (\sigma(a))'$. $\diamondsuit$

A homomorphism $\sigma\colon K \to L$, K, L differential fields, with the property that $\sigma(a') = \sigma(a)'$ for all $a \in K$ is called a *differential homomorphism.* The above corollary shows that conjugate extensions of differential fields are differentially isomorphic.

Example 2.2: Let $a \in K$, $a \neq 0$, K a differential field, n a positive integer not divisible by the characteristic of K. Then

$$(a^{1/n})' = \frac{1}{n}\frac{a^{1/n}}{a}\,a'\,, \quad \text{or} \quad \frac{1}{n}\frac{a'}{a} = \frac{(a^{1/n})'}{a^{1/n}}$$

since $(y^n - a)' = ny^{n-1}y' - a' = 0$ so $y' = a'/(ny^{n-1}) = a'y/(na)$. Notice that in our formulas we always interpret $a^{1/n}$ as one and the same conjugate. A more general formula is the *logarithmic derivative identity* asserting

$$\frac{(a_1^{r_1}\cdots a_n^{r_n})'}{a_1^{r_1}\cdots a_n^{r_n}} = r_1\frac{a_1'}{a_1} + \ldots + r_n\frac{a_n'}{a_n}\,, \quad a_i \in K\,,\ a_i \neq 0\,,\ r_i \in \mathbb{Q}\,,$$

provided that the characteristic of K does not divide the denominators d_i of the r_i and that we are consistent in our interpretation of a_i^{1/d_i} on the left hand side of the identity.

Finally, it should be clear that an algebraic extension of K might contain new constants. For example, $(\mathbb{Q}(x))(y)$ with $y^4 - 2x^2 = 0$ contains either $\sqrt{2}$ or $-\sqrt{2}$ since for $t = y^2/x$, $t^2 = 2$.

Remark: We can now prove that the logarithmic part of the integral of a rational function $\int p/q$ (cf. section 1) is indeed not algebraic. Assume it were, i.e. $y = \int p/q$ where y is an algebraic element over $\mathbb{Q}(x)$. Let N be the normal closure of $(\mathbb{Q}(x))(y)$. Since $\mathrm{char}\,(\mathbb{Q}(x)) = 0$, N is a separable extension of $\mathbb{Q}(x)$ of, say, degree n. Let Γ be the automorphism group of N fixing $\mathbb{Q}(x)$. Then

$$n\frac{p}{q} = \sum_{\sigma\in\Gamma}\sigma(y') = \left(\sum_{\sigma\in\Gamma}\sigma y\right)' = (Trace_{N/\mathbb{Q}(x)}(y))'\,.$$

However, $Trace_{N/\mathbb{Q}(x)}(y) \in \mathbb{Q}(x)$; thus the integral is shown to be rational, which, as we mentioned in section 1, cannot be.

Lemma 2.2: Let K be a differential field and $K(\theta)$ a transcendental extension of K. Then $\theta' = \eta$ induces a derivation on $K(\theta)$ for any $\eta \in K(\theta)$.

Proof: Let $f(\theta) = a_n\theta^n + \ldots + a_0 \in K[\theta]$, $a_n, \ldots, a_0 \in K$. Define $f(\theta)' = a_n'\theta^n + (a_{n-1}' + na_n\eta)\theta^{n-1} + \ldots + (a_0' + a_1\eta)$. As already used in the proof of theorem 2.1, $'$ is a derivation on the ring $K[\theta]$. Since $K(\theta) = \mathrm{QF}(K[\theta])$, lemma 2.1 establishes this lemma. $\diamondsuit$

We now single out various extension of differential fields. In order for several statements made later to be correct we shall from now on assume that *all our domains have characteristic* 0. Also if we write $K \subseteq L$ for two differential fields we shall mean K to be a differential subfield of L, that is, that the derivations of L and K coincide on K.

Lemma 2.3: Let $K \subseteq L$ be differential fields and let $\theta \in L$ such that $\theta' \in K$. Then if there is no element $\eta \in K$, such that $\theta' = \eta'$ then θ is transcendental over K and furthermore $C_{K(\theta)} = C_K$.

Proof: Assume θ is algebraic over K, i.e. there exists a monic irreducible polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = f(x) \in K[x]$ such that $f(\theta) = 0$. Therefore $f(\theta)' = (n\theta' + a_{n-1}')\theta^{n-1} + \ldots = 0$ and, since f was minimal, $n\theta' + a_{n-1}' = 0$, or $\theta' = -a_{n-1}'/n \in K$, contradicting our assumption.

Now we prove that $K(\theta)$ contains no new constants. First, assume $b_n\theta^n + \ldots + b_0 \in K[\theta]$, $b_n \neq 0$, is a constant, i.e. $b_n'\theta^n + (nb_n\theta' + b_{n-1}')\theta^{n-1} + \ldots = 0$. Since θ is transcendental, $b_n' = nb_n\theta' + b_{n-1}' = 0$, hence $\theta' = (-b_{n-1}/nb_n)'$, contradicting our assumption.

Finally, suppose $f(\theta)/g(\theta)$ is a constant, $f, g \in K[\theta]$, $\deg(g) \geq 1$, $\mathrm{GCD}\,(f, g) = 1$, g monic. Thus

$$\left(\frac{f(\theta)}{g(\theta)}\right)' = \frac{f(\theta)'}{g(\theta)} - \frac{f(\theta)g(\theta)'}{g^2(\theta)} = 0\,,$$

or $f(\theta)/g(\theta) = f(\theta)'/g(\theta)'$. But $\deg(g(\theta)') < \deg(g(\theta))$ which is impossible since f/g was already in lowest terms. $\diamondsuit$

We wish to remark that using the previous lemma we once again have established that the logarithmic part of the integral of a rational function is transcendental.

Let $K \subseteq L$ be differential fields. If for $\theta \in L$ there exists an element $\eta \in K$ such that $\theta' = \eta$ we call the extension $K(\theta)$ an extension of K by an *integral*, and call θ *primitive* over K. If $\theta' = \eta'/\eta$, $\eta \in K$, $\eta \neq 0$, then we call $K(\theta)$ an extension of K by a *logarithm* and write $\theta = \log \eta$. Obviously, extensions by logarithms are extensions by integrals.

If for $\theta \in L$ there exists an element $\eta \in K$ such that $\theta'/\theta = \eta$, $\theta \neq 0$, we call $K(\theta)$ an extension of K by an *exponential of an integral.* Furthermore, if there exists an element $\xi \in K$ such that $\xi' = \eta$ then we call $K(\theta)$ an extension of K by an *exponential* and write $\theta = \exp \xi$. Obviously, extensions by exponentials are extensions by exponentials of integrals.

Definition 1.2: Let $K \subseteq L$ be differential fields, $\theta \in L$. $K(\theta)$ is a *simple elementary extension of* K if θ is algebraic over K, or $\theta = \log \eta$, or $\theta = \exp \eta$, $\eta \in K$; θ is called a *monomial* over K if $\theta = \log \eta$ or $\theta = \exp \eta$, $\eta \in K$, and θ is transcendental over K as well as $C_K = C_{K(\theta)}$.

L is an *elementary (generalized) Liouville extension* of K if there exist differential fields $K = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = L$ such that for all $1 \leq i \leq n$, F_i is a simple elementary extension (an extension by an algebraic element, or an integral, or an exponential of an integral) of F_{i-1}. L is a *regular* elementary Liouville extension of K if the intermediate non-algebraic extensions are extensions by monomials.

Obviously, elementary Liouville extensions are generalized ones. Also a regular elementary Liouville extension L of K cannot contain new transcendental constants, i.e. $C_L \subset \bar{C_K}$, the algebraic closure of C_K. The latter inclusion requires proof, so let d be algebraic over the differential field K with $d' = 0$. Assume that $d^n + a_{n-1}d^{n-1} + \ldots + a_0 = 0$, $a_i \in K$, is the minimal polynomial of d. By theorem 2.1 it follows that $a'_{n-1}d^{n-1} + \ldots + a'_0 = 0$, which implies that $a'_{n-1} = \ldots = a'_0 = 0$ because of the minimality for n. Therefore any constant algebraic over K is in fact algebraic over C_K.

Example 2.3: We shall take the liberty of nesting extensions by simply listing them, that is $K(\exp \eta_1, \log \eta_2) = (K(\exp \eta_1))(\log \eta_2)$.

1. $\mathbb{Q}(x, \exp(x), \log(\exp(x) + 1), \exp(x)^{2/3})$ is a regular elementary Liouville extension of $\mathbb{Q}(x)$, though we cannot prove this yet.
2. $\mathbb{Q}(x, \exp(x), \exp(2x + 1))$ is an elementary Liouville extension over $\mathbb{Q}(x)$, but it is not regular since $\exp(2x + 1)/\exp(x)^2 = \exp(1)$ and thus introduces a new transcendental constant.
3. $\mathbb{Q}(x, \log(x), \exp(\log(x)/3))$ is not an extension by a monomial of $\mathbb{Q}(x, \log(x))$, because $\exp(\log(x)/3) = x^{1/3}$ is algebraic over this field.

Remark: Generalized Liouville extensions play a role in solving linear differential equations of the form

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_0 = 0\,, \quad a_i \in K$$

in closed form, where $K = \mathbb{Q}(x)$ or a finite algebraic extensions of it. This problem leads to a well-developed Galois theory, but we shall not go into further detail here. (Cf. E. Kolchin [53], M. Singer [82].)

The last item in the previous example indicates that adjoining exponentials and logarithms might possibly cause an inconsistency with respect to the derivation. For if $\theta = \exp(\eta)$, $\eta = \log(x)/3$, how could we have been sure that $\theta' = \theta\eta' = 1/(3x^{2/3})$. We need the following important lemma:

Lemma 2.4: Let $L \subseteq K$ be differential fields, $\theta \in L$ such that θ is *not* a monomial over K.

a) If $\theta = \log \eta$, $\eta \in K$, then there exist $\xi \in K$, $c \in C_L$ such that $\theta = \xi + c$, i.e. $\xi' = \eta'/\eta$.

b) If $\theta = \exp \eta$, $\eta \in K$, then there exist $\xi \in K$, $c \in C_L$ and an integer $n > 0$ such that $\theta^n = c\xi$, i.e. $\xi'/\xi = n\eta'$.

Proof: a) By lemma 2.3, $\theta = \int \eta'/\eta$ is not a monomial if there exists a $\xi \in K$ such that $\xi' = \eta'/\eta$. Let $c = \theta - \xi$, then obviously $c' = 0$.

b) We claim that there exists $d \in C_L$ such that θ is algebraic over $K(d)$. If θ is algebraic over

K itself, the choice $d = 0$ works. So suppose that θ is transcendental over K. Since θ is not a monomial there must exist a constant $d = f(\theta)/g(\theta)$, $f, g \in K[\theta]$, $g \neq 0$, with $d \notin K$. Thus $f(\theta) - dg(\theta) = 0$, meaning that θ is algebraic over $K(d)$ and therefore d is transcendental over K by the assumption of the transcendence for θ. Now let N be the normal closure of $K(\theta)$ over $K(d)$, $n = [N : K(d)]$. Then for Γ being the automorphism group on N fixing $K(d)$

$$n\eta' = \sum_{\sigma\in\Gamma} \sigma\left(\frac{\theta'}{\theta}\right) = \frac{\left(\prod_{\sigma\in\Gamma}\sigma(\theta)\right)'}{\prod_{\sigma\in\Gamma}\sigma(\theta)} = \frac{(Norm_{N/K(d)}(\theta))'}{Norm_{N/K(d)}(\theta)}\,.$$

Setting $\xi(d) = Norm_{N/K(d)}(\theta) \in K(d)$ we have shown that

$$n\eta'\xi(d) - \xi(d)' = 0\,, \quad \xi(d) \neq 0\,.$$

We now prove that we can actually find $\xi \in K$, $\xi \neq 0$, solving the same differential equation. Obviously, we only need to consider the case $d \neq 0$, which is d being transcendental over K. Let $\xi(d) = f(d)/g(d)$, $f, g \in K[d]$, $g \neq 0$. Then

$$h(d) = n\eta' f(d)g(d) - f(d)'g(d) + f(d)g(d)' = 0\,,$$

where $h(d) \in K[d]$. Since d is transcendental, $h = 0$ and we choose $d = i \in \mathbb{Z}$ such that $f(i)g(i) \neq 0$. Then $n\eta'\xi(i) - \xi(i)' = 0$ with $\xi(i) \in K$, $\xi(i) \neq 0$. Therefore

$$\left(\frac{\theta^n}{\xi}\right)' = \frac{n\theta^{n-1}\theta'}{\xi} - \frac{\theta^n\xi'}{\xi^2} = \frac{n\theta^n\eta'}{\xi} - \frac{\theta^n n\eta'}{\xi} = 0$$

hence $\theta^n = c\xi$ for some constant $c \in C_L$. $\diamondsuit$

This lemma, from a theoretical point of view, removes the a-priori need for our extension field L.

Corollary: Let K be a differential field with an algebraically closed field of constants and let $0 \neq \eta \in K$. Then there exists a differential field $L \supseteq K$ and a $u \in L$ such that

$$C_L = C_K \quad \text{and} \quad u' - u\eta' = 0 \qquad (\text{or} u' - \eta'/\eta = 0)\,.$$

Proof: If there exist $\xi \in K$, $n \in \mathbb{Z}$, $n > 0$, such that $\xi'/\xi = n\eta'$ then we choose L the algebraic extension of K by y satisfying $y^n - \xi = 0$. Since C_K is algebraically closed, $C_{K(y)} = C_K$. Furthermore

$$y' = \frac{\xi'}{ny^{n-1}} = \frac{y\xi'}{n\xi} = y\eta'\,,$$

which proves that y is a solution to our differential equation. In case no such pair ξ, n exists we choose L the transcendental extension $K(\theta)$ with $\theta' = \theta\eta'$. By the previous lemma, $C_{K(\theta)} = C_K$, for otherwise θ would not be a monomial meaning that $\theta^n = c\xi$ resulting in a pair xi, n with the above properties. The statement for the differential equation $u' - \eta'/\eta$ is an easy consequence of lemma 2.3. $\diamondsuit$

Any minimal field having the property of the field L constructed in the above proof is differentially isomorphic to L. L is called the Picard-Vessiot extension of K with respect to the given differential equation. Thus the notions $K(\exp\eta)$ and $K(\log\eta)$ are consistent insofar that we associate with them the fields L constructed in our corollary. This notion is still somewhat unsatisfactory since it makes $K(\log\eta, \log 2\eta) = K(\log\eta)$.

Now we discuss how algebraic dependencies can be discovered in elementary Liouville extensions. As we have seen before, exponentials may hide truly algebraic extensions, e.g. in $K(\log\lambda, \exp\eta)$[†]

$$\exp(c_1 + r_1 \log\lambda + r_2\eta) = c_2\lambda^{r_1}\exp(\eta)^{r_2}$$

and

$$\log(c_1\lambda^{r_1}\exp(\eta)^{r_2}) = c_2 + r_1\log\lambda + r_2\eta\,,$$

where $r_1, r_2 \in \mathbb{Q}$, c_1, c_2 constants. These are, of course, well-known algebraic relations for logarithms and exponentials but we will prove that they are the only ones possible. Once this *Structure Theorem* is established we can easily discover whether a logarithm or exponential is a monomial, but again encounter the ubiquitous problem of constants.

Theorem 2.2: For $1 \le i \le n$ let $K_i = C(x, \theta_1, \ldots, \theta_i)$ be an elementary Liouville extension of $K_0 = C(x)$ and assume that $C_{K_n} = C$. Let

$$E = \{i \mid \theta_i = \exp(\eta_i)\,,\ \eta_i \in K_{i-1}\,,\ \theta_i \text{ a monomial}\,,\ 1 \le i \le n\}$$
$$\Lambda = \{j \mid \theta_j = \log(\lambda_j)\,,\ \lambda_j \in K_{j-1}\,,\ \theta_j \text{ a monomial}\,,\ 1 \le j \le n\}$$

If $u, v \in K_n$ such that $u'/u = v'$ then there exist rational numbers $r_i, i \in E \cup \Lambda$, and a constant $c \in C$ such that

$$v = c + \sum_{i\in E} r_i\eta_i + \sum_{j\in\Lambda} r_j\theta_j\,. \quad \diamondsuit$$

We shall not prove this theorem in its entire generality, but refer the reader to Risch [79] and Rothstein and Caviness [79]. Unfortunately, neither presentation appears to us self-contained. However, we will supply a proof for the case in which no θ_i is algebraic over K_{i-1} in section 3. Since we will apply only this case to the intgration problem, our omission of a proof for the general theorem will not effect any of the latter theorems or algorithms. Now we shall draw an immediate consequence of this theorem:

Theorem 2.3 *(Structure Theorem):* Let $C, K_i\,,\ 0 \le i \le n$, E and Λ be as in theorem 2.2.

a) If $K_n(\theta_{n+1})$ is an extension of K_n by an exponential $\theta'_{n+1} = \theta_{n+1}\eta'_{n+1}$, $\eta_{n+1} \in K_n$, and if θ_{n+1} is not a monomial then there exist rational numbers $r_i, i \in E \cup \Lambda$ and a constant $c \in C$ such that

$$\eta_{n+1} = c + \sum_{i\in E} r_i\eta_i + \sum_{j\in\Lambda} r_j\theta_j$$

b) If $K_n(\theta_{n+1})$ is an extension of K_n by a logarithm $\theta'_{n+1} = \lambda'_{n+1}/\lambda_{n+1}$, $\lambda_{n+1} \in K_n$, and if θ_{n+1} is not a monomial then there exist integers $r, r_i, i \in E \cup \Lambda$, $r \ne 0$ and a constant

† From now to the end of this section we shall, for the sake of clarity, denote arguments to exps by *eta*, subscripted by i if necessary, and arguments to logs by *lambda*, subscripted by j if necessary.

$c \in C$ such that

$$\lambda_{n+1}^r = c \left(\prod_{i\in E} \theta_i^{r_i}\right)\left(\prod_{j\in\Lambda} \lambda_j^{r_j}\right).$$

Proof: We shall show that the previous theorem implies this one.

a) By lemma 2.4 there exist $\xi \in K_n$, $k \in \mathbb{Z}$, $k > 0$ such that $\xi'/\xi = (k\eta_{n+1})'$. Applying theorem 2.2 to $u = \xi$, $v = k\eta_{n+1}$ we get

$$k\eta_{n+1} = c + \sum_{i\in E} r_i\eta_i + \sum_{j\in\Lambda} r_j\theta_j\,.$$

Dividing by k yields this part.

b) By lemma 2.4 there exists $\xi \in K_n$ such that $\xi' = \lambda_{n+1}'/\lambda_{n+1}$. Applying theorem 2.2 to $u = \lambda_{n+1}$, $v = \xi$ we get, by removing denominators,

$$r\xi = c + \sum_{i\in E} r_i\eta_i + \sum_{j\in\Lambda} r_j\theta_j\,, \quad r, r_i, r_j \in \mathbb{Z}\,, \quad c \in C\,, \quad r \neq 0\,.$$

Let $P = (\prod_{i\in E}\theta_i^{r_i})(\prod_{j\in\Lambda}\lambda_j^{r_j})$ with r_i, r_j from above. Then

$$\frac{P'}{P} = \sum_{i\in E} r_i\frac{\theta_i'}{\theta_i} + \sum_{j\in\Lambda} r_j\frac{\lambda_j'}{\lambda_j} = \sum_{i\in E} r_i\eta_i' + \sum_{j\in\Lambda} r_j\theta_j' = r\xi'$$

hence

$$\left(\frac{\lambda_{n+1}^r}{P}\right)' = \frac{r\lambda_{n+1}^{r-1}\lambda_{n+1}'}{P} - \frac{P'\lambda_{n+1}^r}{P^2} = \frac{r\lambda_{n+1}^r\xi'}{P} - \frac{r\xi'\lambda_{n+1}^r}{P} = 0$$

showing that $\lambda_{n+1}^r/P \in C$. ◊

Example 2.3 (continued): We can now show that $\log(\exp(x) + 1)$ is a monomial over $\mathbb{Q}(x, \exp(x))$. First of all, $\exp(x)$ is a monomial since $\theta = \exp(x)$ is not a constant ($E\cup\Lambda = \emptyset$). Now assume that $\log(\theta + 1)$ is not a monomial. By the structure theorem $(\theta + 1)^r = c\theta^{r_1}$ for some integers $r, r_1 > 0$ and some constant c. But this is clearly impossible.

We now give an algorithm which decides whether θ_{n+1} is a monomial over K_n under the assumption that K_n is a regular elementary *purely transcendental* Liouville extension over $C(x)$. Let $\bar\Lambda = \Lambda \cup \{n + 1\}$ if $\theta_{n+1} = \log\lambda_{n+1}$ and let $\bar E = E \cup \{n + 1\}$ if $\theta_{n+1} = \exp\eta_{n+1}$, otherwise let $\bar\Lambda = \Lambda$ and $\bar E = E$. Now write

$$\eta_i = \frac{p_i(\theta_1, \ldots, \theta_{i-1})}{q_i(\theta_1, \ldots, \theta_{i-1})}\,, \quad i \in \bar E\,, \qquad \lambda_j = \frac{p_j(\theta_i, \ldots, \theta_{j-1})}{q_j(\theta_1, \ldots, \theta_{j-1})}\,, \quad j \in \bar\Lambda\,,$$

where p_k, $q_k \in C[x, \theta_1, \ldots, \theta_{k-1}]$ for $k \in \bar\Lambda \cup \bar E$.

Case $\theta_{n+1} = \exp(\eta_{n+1})$: If θ_{n+1} is not a monomial we can solve

$$\frac{p_{n+1}}{q_{n+1}} = c + \sum_{i\in E} r_i\frac{p_i}{q_i} + \sum_{j\in\Lambda} r_j\theta_j$$

in rationals r_i, r_j and a constant c. Multiplying by the common denominator and comparing coefficients of power products of x, $\theta_1, \ldots, \theta_n$ leads to a linear system in c, r_i, r_j. If this system has no solution with $c \in C$, $r_i, r_j \in \mathbb{Q}$ then θ_{n+1} must be a monomial. (Notice that for $C = \mathbb{Q}$ these conditions can be effectively tested.) If a solution $c \in C$, $r_i, r_j \in \mathbb{Q}$ can be found θ is not a monomial.

Case $\theta_{n+1} = \log(\lambda_{n+1})$: Find squarefree and pairwise relatively prime polynomials $b_1, \ldots, b_l \in C[x, \theta_1, \ldots, \theta_n]$ such that each $p_j, q_j, j \in \bar{\Lambda}$, and $\theta_i, i \in E$, can be expressed as a power product of the b_k's times a constant. Notice that all $\theta_i, i \in E$, occur among the b_i's. Let

$$\theta_i = \prod_{k=1}^{l} b_k^{e_{ik}}\,, \quad i \in E\,, \quad e_{ik} = 0 \text{ for all } k \text{ except one which is } 1$$

and

$$\lambda_j = c_j \prod_{k=1}^{l} b_k^{e_{jk}}\,, \quad j \in \bar{\Lambda}\,, \quad e_{jk} \in \mathbb{Z}\,, \quad c_j \in C\,.$$

If θ_{n+1} is not a monomial we can solve

$$c_{n+1}^r \prod_{k=1}^{l} b_k^{re_{n+1,k}} = c \prod_{j \in \Lambda} c_j \prod_{k=1}^{l} b_k^{\left(\sum_{m \in E \cup \Lambda} r_m e_{mk}\right)}$$

for a constant c and integers r, r_i, r_j. There exists a solution only if

$$\sum_{m \in E \cup \Lambda} r_m e_{mk} = r e_{n+1,k} \quad \text{for } k = 1, \ldots, l$$

which is a homogeneous linear system with integral coefficients. If the system has a non-trivial solution θ_{n+1} is not a monomial, otherwise it is one. Another method in this case is to take the logarithm of $\lambda_{n+1}^r = c \prod_{i \in E} \theta_i^{r_i} \prod_{j \in \Lambda} \lambda_j^{r_j}$, differentiate and solve

$$r\,\frac{\lambda'_{n+1}}{\lambda_{n+1}} = \sum_{i \in E} r_i \eta'_i + \sum_{j \in \Lambda} r_j \frac{\lambda'_j}{\lambda_j}$$

as we did in the exponential case.

Example 2.4: $\log(x \exp(x)) + \exp(\exp(x) + \log(x))$. We want to find a regular elementary purely transcendental Liouville extension of $\mathbb{Q}(x)$ in which this function lies. We chose $\theta_1 = \exp(x)$ which is a monomial by our previous argument. Next we investigate $\theta_2 = \log(x\theta_1)$: $\lambda_2 = x\theta_1$, hence $b_1 = x$, $b_2 = \theta_1$ build a basis for θ_1, λ_2. This leads us to

$$b_1^r b_2^r = c b_1^0 b_2^{r_1}$$

whose only solution is $r = r_1 = 0$. Thus θ_2 is a monomial over $\mathbb{Q}(x, \theta_1)$. The next extension we choose is $\theta_3 = \log x$: $\lambda_3 = x$, hence $b_1 = x$, $b_2 = \theta_1$ is again a basis for θ_1, λ_2, λ_3. But now the equation

$$b_1^r b_2^0 = c(b_1 b_2)^{r_1} b_2^{r_2} = c b_1^{r_1} b_2^{r_1 + r_2}$$

has the solution $r = r_1 = 1$, $r_2 = -1$, and $c = 1$. Hence θ_3 is not a monomial, since $x = \lambda_2/\theta_1$ or $\theta_3 = \theta_2 - x+$ a constant. Assume that this constant is 0. Then $\theta_3 \in \mathbb{Q}(x, \theta_1, \theta_2)$ and we still have a regular elementary Liouville extension. Finally we consider $\theta_1 = \exp(\theta_1 + \theta_2 - x)$. This leads to the equation

$$\theta_1 + \theta_2 - x = c + r_1 x + r_2 \theta_2$$

which by comparing the coefficient of θ_1, $1 = 0$, has no solution. Therefore θ_4 is a monomial over $\mathbb{Q}(x, \theta_1, \theta_2)$ and our expression lies in the regular elementary Liouville extension $\mathbb{Q}(x, \theta_1, \theta_2, \theta_4)$. The assumption on the constant $\log(1) = 0$ depends, of course, on the branch on which both occurring logs lie.

There are several remarks in order on finding regular elementary Liouville extensions for a given expression.

1. We want to imbed our integrand in a regular extension because this is the data structure on which the later algorithm will work. Also regular extensions are stable with respect to different interpretations of the log function. In fact, the maps $\sigma: K_n \to K_n$, $\sigma(\theta_j) = \theta_j + 2\pi i n_j$, $j \in \Lambda$, $n_j \in \mathbb{Z}$, $\sigma(x) = x$, $\sigma(\theta_i) = \theta_i$, $i \in E$ are differential isomorphisms on a suitable domain of the complex plane (cf. Risch [1969], Proposition 1.3).
2. The order by which logs and exps are adjoined is important. First of all, an appearing algebraic might not be one if we choose a different order. E.g. $\exp(x)$ is algebraic over $\mathbb{Q}(x, \exp(2x))$ whereas $\exp(2x) \in \mathbb{Q}(x, \exp(x))$. It is not even clear to us whether a monomial can be discovered by just scanning the expression. E.g. $\exp(2x) + \exp(3x) \in \mathbb{Q}(\exp(x))$ but neither of the terms in the expression will produce a regular extension. Secondly, the integration algorithm may discover very quickly, that a given integral has no closed form solution by using one particular tower of fields whereas for another tower of fields it may take quite long. We know of no reference mentioning or studying this phenomenon.
3. *The Problem of Constants:* A new exponential or logarithm may fail to be in the previous field by virtue of just a constant, e.g. $\exp(x + 1)$ over $\mathbb{Q}(x, \exp(x))$. It may appear reasonable to enlarge the constant field to $\mathbb{Q}(\exp(1))$. The problem is that another such adjoined constant (e.g. $2\pi i$), may be algebraic over this field without our knowledge. This, of course, means that we cannot affirmatively decide whether a constant is zero or not. Our integration algorithm might then output a closed form solution which is none since one of the produced denominators vanishes. Various theorems and conjectures may help tackle particular cases:
   - (Lindemann [1882])
     If $a_1, \ldots, a_n \in \bar{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$ the transcendence degree
     $$\text{tr deg}(\mathbb{Q}(\exp a_1, \ldots, \exp a_n)/\mathbb{Q}) = n\,.$$
   - *Schanuel's conjecture* (cf. Lang [71]):
     If $c_1, \ldots, c_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ then
     $$\text{tr deg}(\mathbb{Q}(c_1, \ldots, c_n, \exp(c_1), \ldots, \exp(c_n))/\mathbb{Q}) \geq n\,.$$

This conjecture implies that e and π are algebraically independent, but even this has not been established.

Historic Remarks: The invention of differential algebra is usually attributed to E. Kolchin [42] and R. Ritt [50]. The Structure Theorem is due to R. Risch [79]. The algorithm for deciding whether a logarithm is a monomial is due to H. Epstein [76]. Example 2.4 is taken from B. Caviness [77]. The connection between Schanuel's conjecture and the problem of constants is discussed in B. Caviness and M. Prelle [78].

3. Liouville's Theorem

We shall state the "strong" version of this theorem right away and then slowly establish the necessary lemmas to prove it.

Theorem 3.1 *(Strong Liouville Theorem):* Let L be an elementary Liouville extension of the differential field K and let $\bar{C}$ be the algebraic closure of C_K. Assume there is an element $g \in L$ with $g' \in K$. Then there exist elements $v_0 \in K$, $\bar{v}_1, \ldots, \bar{v}_n \in \bar{C}K$, $c_1, \ldots, c_n \in \bar{C}$ such that

$$g' = v_0' + c_1 \frac{\bar{v}_1'}{\bar{v}_1} + \ldots + c_n \frac{\bar{v}_n'}{\bar{v}_n}. \quad \diamondsuit$$

The importance of this theorem to the problem of integration is obvious. Instead of looking at any possible elementary Liouville extension of the field of our integrand, we essentially only need to adjoin logarithms.

Lemma 3.1: Let $K(\theta) \supset K$ be differential fields, $C_K = C_{K(\theta)}$ and θ transcendental over K.

a) If $\theta' \in K$ then for any polynomial $p(\theta) \in K[\theta]$, $p(\theta)'$ as a polynomial in $K[\theta]$ has either the same degree as p, or degree one less, the latter exactly if $\mathrm{ldcf}_\theta(p(\theta))$ is a constant.

b) If $\theta'/\theta \in K$ then for any non-constant polynomial $p(\theta) \in K[\theta]$, $p(\theta)' \in K[\theta]$ has the same degree in θ as p.

Proof: a) Let $p(\theta) = a_n\theta^n + \ldots + a_0$, $a_i \in K$, $a_n \neq 0$, $n > 0$. Then

$$p(\theta)' = a_n'\theta^n + (na_n\theta' + a_{n-1}')\theta^{n-1} + \ldots .$$

If $a_n' = 0$ and $na_n\theta' + a_{n-1}' = 0$ then $(na_n\theta + a_{n-1})' = 0$, hence $na_n\theta + a_{n-1}$ is a new constant in $K(\theta) \setminus K$, contradicting our assumption $C_{K(\theta)} = C_K$.

b) Let $\theta' = \eta\theta$. For $a\theta^n$, $a \in K$, $a \neq 0$, $n > 0$, we get

$$(a\theta^n)' = (a' + na\eta)\theta^n .$$

Were $a' + na\eta = 0$ then $a\theta^n$ would be a new constant in $K(\theta) \setminus K$. From this our degree condition follows immediately. $\diamondsuit$

Lemma 3.2: Let $K(\theta) \supset K$ be differential fields, $C_K = C_{K(\theta)}$, and θ transcendental over K. Furthermore, let $p(\theta) \in K[\theta]$ be monic with $\mathrm{GCD}_\theta(p(\theta),\, dp(\theta)/d\theta) = 1$.

a) If $\theta' \in K$ then $\mathrm{GCD}_\theta(p(\theta), p(\theta)') = 1$.

b) If $\theta'/\theta \in K$ then $\mathrm{GCD}_\theta(p(\theta), p(\theta)') = 1$ unless $\theta \mid p$.

Proof: We first consider irreducible p. In the case that $\theta' \in K$ we have, by lemma 3.1, that $\deg_\theta(p') < \deg_\theta(p)$, hence part a) is verified for irreducible p. Now let $\theta' = \theta\eta$ and let $p(\theta) = \theta^n + a_{n-1}\theta^{n-1} + \ldots + a_0$, $a_i \in K$, $n > 0$. Assume further that $\theta \nmid p$, i.e. $a_0 \neq 0$. By lemma 3.1, $p(\theta)' = n\eta\theta^n + \ldots + a_0'$ has the same degree in θ as p. Suppose that $\mathrm{GCD}\,(p(\theta), p(\theta)') \neq 1$. From the irreducibility of p we get that $p(\theta)'$ is a multiple of p by the factor $n\eta$, hence $n\eta a_0 = a_0'$. Therefore, $(\theta^n/a_0)' = 0$ meaning that θ^n/a_0 is a new constant in $K(\theta) \setminus K$. This contradicts the assumption made.

We finally treat the case that p is composite. Let $p = qg$ such that $g \mid p'$ and g is irreducible. Since $\mathrm{GCD}\,(p(\theta), dp(\theta)/d\theta) = 1$ we have $\mathrm{GCD}\,(g, q) = 1$. But $p(\theta)' = q(\theta)'g(\theta) + q(\theta)g(\theta)'$ hence $g(\theta)$ must divide $g(\theta)'$. From the above we know that this is only possible if θ is the exponential of an integral and $\theta \mid g$ that is $\theta \mid p$. $\diamondsuit$

Theorem 3.2 (*Weak Liouville Theorem*): Let L be an elementary Liouville extension of the differential field K with $C_K = C_L$. Assume that there is an element $g \in L$ with $g' \in K$. Then there exist elements $v_0, v_1, \ldots, v_n \in K$, $c_1, \ldots, c_n \in C_K$ such that

$$g' = v_0' + c_1 \frac{v_1'}{v_1} + \ldots + c_n \frac{v_n'}{v_n}$$

Proof: Let $L = K(\theta_1, \ldots, \theta_m)$ be the elementary extension. The proof proceeds by induction on m. For $m = 0$, $g \in K$ and hence $v_0 = g$. Assume the theorem is true for $m-1$ and arbitrary fields K. Now let $g \in K(\theta_1, \ldots, \theta_m) = (K(\theta_1))(\theta_2, \ldots, \theta_m)$. By hypothesis there exist $v_0, \ldots, v_n \in K(\theta_1)$, $c_1, \ldots, c_n \in C_K$ such that

$$g' = v_0' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}.$$

From now on we write θ for θ_1.

Case 1: θ is algebraic over K. Let N be the normal closure of $K(\theta)$, Γ the automorphism group on N fixing K, l the cardinality of Γ. Then

$$lg' = \sum_{\sigma \in \Gamma} \left(\sigma v_0' + \sum_{i=1}^{n} c_i \frac{\sigma v_i'}{\sigma v_i} \right) = \left(\sum_{\sigma \in \Gamma} \sigma v_0 \right)' + \sum_{i=1}^{n} c_i \frac{\left(\prod_{\sigma \in \Gamma} \sigma v_i\right)'}{\prod_{\sigma \in \Gamma} \sigma v_i},$$

by the logarithmic derivative identity, and since $w_0 = \sum_\sigma \sigma v_0 / l$ and $w_i = \prod_\sigma \sigma v_i$ are elements in K we have the desired form

$$g' = w_0' + \sum_{i=1}^{n} \frac{c_i}{l} \frac{w_i'}{w_i}.$$

Case 2: θ is transcendental over K. Using

$$\frac{(a/b)'}{a/b} = \frac{a'}{a} - \frac{b'}{b}$$

we can find polynomials $u_j \in K[\theta]$, $1 \leq j \leq k$ such that

$$\sum_{i=1}^{n} c_i \frac{v_i'}{v_i} = \sum_{j=1}^{k} d_j \frac{u_j'}{u_j}$$

with $d_j \in C_K$, u_j either in K or monic, squarefree, pairwise relatively prime polynomials. We can also find $w_0, w_{ij}, f_i \in K[\theta]$, $1 \le i \le r$, $1 \le j \le i$ such that

$$v_0 = w_0 + \sum_{i=1}^{r} \sum_{j=1}^{i} \frac{w_{ij}}{f_i^j}$$

with $\deg_\theta(w_{ij}) < \deg_\theta(f_i)$, f_i monic, squarefree and pairwise relatively prime. The f_i, w_{ij}, and u_j have to be of a very special form to give $g' \in K$ on the left hand side of the above equation. We now separate cases.

Case 2.1: $\theta = \log \eta$. Using lemma 3.2 we prove similarly as in lemma 1.1 that $w_{ij} = 0$ for all $1 \le i \le r$, $1 \le j \le i$. For otherwise, the partial fraction decomposition of v_0' would contain a denominator of multiplicity ≥ 2 which could not cancel. This also implies that none of the u_j are polynomials. Finally, by lemma 3.1, $w_0 = c\theta + a$, $c \in C_K$, $a \in K$. Therefore

$$g' = a' + c\frac{\eta'}{\eta} + \sum_{i=1}^{k} d_i \frac{u_i'}{u_i}$$

which is the desired form.

Case 2.2: $\theta = \exp \eta$. We choose $u_1 = f_{i_0} = \theta$. Then as argued above, $w_{ij} = 0$ for $i \ne i_0$, $1 \le j \le i$, and $u_j \in K$ for $2 \le j \le k$. Since

$$\left(\frac{a}{\theta^j}\right)' = \frac{a' - aj\eta'}{\theta^j} \ne 0\,, \quad a \in K\,,\ a \ne 0\,, \quad \text{and } \frac{u_1'}{u_1} = \eta'\,,$$

we conclude as before that $w_{i_0 j} = 0$. Also, by lemma 3.1, $w_0 \in K$. Therefore

$$g' = (w_0 + d_1\eta)' + \sum_{i=2}^{k} d_i \frac{u_i'}{u_i}$$

which is the desired form. $\diamondsuit$

Proof of theorem 2.2: See appendix.

From theorem 3.2 we can already conclude that there is no elementary Liouville extension of $\mathbb{C}(x, \exp(x^2))$ containing the integral $\int \exp(x^2)\,dx$. We answer, more generally, under which conditions $\int g(x)\exp(f(x))\,dx$, $f(x), g(x) \in \mathbb{C}(x)$, $f(x)$ non-constant, can be found. Let $\theta = \exp(f(x))$. Now θ is transcendental over $\mathbb{C}(x)$ by the structure theorem 2.2 (or already by lemma 2.4). Assume we can locate $\int g(x)\exp(f(x))\,dx$ in an elementary Liouville extension of $\mathbb{C}(x,\theta)$. By theorem 3.2 there exist $v_0, \ldots, v_n \in \mathbb{C}(x,\theta)$, $c_1, \ldots, c_n \in \mathbb{C}$, such that

$$g(x)\theta = v_0(x,\theta)' + \sum_{i=1}^{n} c_i \frac{v_i(x,\theta)'}{v_i(x,\theta)}\,.$$

We now interpret $v_i \in F(\theta)$ with $F = \mathbb{C}(x)$. As in the proof of theorem 3.2 we conclude that $\sum_{i=1}^{n} c_i v_i'/v_i \in F$ and $v_0 = y\theta + z$, $y, z \in F$. Therefore

$$g\theta = (y' + yf')\theta \quad \text{or} \quad g = y' + yf'\,.$$

In this case $\int g \exp(f) = y \exp(f)$.

Now it is easily shown that $1 = y' + 2xy$ has no solution in $\mathbb{C}(x)$. Similarly, $1/x = y' + y$ has no solution in $\mathbb{C}(x)$, hence $\int \exp(x)/x\,dx$ is non-elementary. substituting $\exp(t)$ for x we get $\int \exp(\exp(t))\,dt$ non-elementary, substituting $\log(t)$ for x we get $\int dt/\log(t)$ non-elementary. Finally

$$\int \log\log x\,dx = x\log\log x - \int \frac{dx}{\log x}$$

is not elementary.

In fact, there is an algorithm to decide whether $y' + f'y = g$, $f, g \in \mathbb{C}(x)$, has a solution $y \in \mathbb{C}(x)$. For later reference we shall formulate the following theorem much more general than we need at this moment.

Theorem 3.3: Let $C(x)$ be the transcendental extension of the constant field C with $x' = 1$. Assume that $f, g_1, \ldots, g_m \in C(x)$ are given. Consider the differential equation

$$y' + fy = \sum_{i=1}^{m} c_i g_i \quad \text{with} \quad c_1, \ldots, c_n \in C \tag{$*$}$$

in $y \in C(x)$. Then we can compute, in a finite number of arithmetic operations in C (including computation of integral roots of polynomials in $C[z]$), elements $h_1, \ldots, h_r \in C(x)$ and a linear system S with coefficients in C and variables $z_1, \ldots, z_r$, $e_1, \ldots, e_m$ such that

y solves $(*)$ if and only if

$$y = \sum_{i=1}^{r} y_i h_i \quad \text{and} \quad z_1 = y_1, \ldots, z_r = y_r\,,\ e_1 = c_1, \ldots, e_m = c_m \quad \text{solve } S\,.$$

Proof: We represent, by GCD computations,

$$f(x) = \frac{p(x)}{q_1(x)^{k_1} \cdots q_n(x)^{k_n}}\,, \qquad \sum_{i=1}^{m} c_i g_i(x) = \frac{G(x)}{q_1(x)^{l_1} \cdots q_n(x)^{l_n}}$$

where $p, q_1, \ldots, q_n \in C[x]$, $G \in C[c_1, \ldots, c_m, x]$, $q_1, \ldots, q_n$ monic, squarefree and pairwise relatively prime, $\mathrm{GCD}\,(p, q_i) = 1$, $k_i \geq 0$, $l_i \geq 0$ for $1 \leq i \leq n$. Arguing as in the proof of theorem 3.2 we conclude that if $y(x)$ solves $(*)$ then

$$y(x) = \frac{Y(x)}{q_1(x)^{j_1} \cdots q_n(x)^{j_n}}$$

with $Y(x) \in C[c_1, \ldots, c_m, x]$, $j_i \geq 0$ for $1 \leq i \leq n$. We first compute a bound $\bar{j}_i$ for j_i, $1 \leq i \leq n$, and then a bound $\bar{\alpha}$ for $\deg_x(Y)$. Let

$$y(x) = \frac{A_{i,j_i}(x)}{q_i(x)^{j_i}} + \ \ldots\,, \qquad f(x) = \frac{b_{i,k_i}(x)}{q_i(x)^{k_i}} + \ \ldots\,, \qquad \sum_{s=1}^{m} c_s g_s(x) = \frac{D_{i,l_i}(x)}{q_i(x)^{l_i}} + \ \ldots\,.$$

be the partial fraction decompositions of y, f, and $\sum c_s g_s$ with $A_{i,j_i}, D_{i,l_i} \in C[c_1,\ldots,c_m,x]$, $b_{i,k_i} \in C[x]$ and $\deg_x(A_{i,j_i}) < \deg(q_i)$ unless $j_i = 0$, $\deg(b_{i,k_i}) < \deg(q_i)$ unless $k_i = 0$. Substituting these expansions into $(*)$ we get

$$-\frac{j_i q_i' A_{i,j_i}}{q_i^{j_i+1}} + \ldots + \frac{b_{i,k_i} A_{i,j_i}}{q_i^{j_i+k_i}} + \ldots = \frac{D_{i,l_i}}{q_i^{l_i}} + \ldots \quad .$$

We first observe that $j_i + 1 \le l_i$ is equivalent to $j_i + k_i \le l_i$ since otherwise one of the leading terms could not cancel on the left-hand side. The third possibility is that $j_i + 1 = j_i + k_i > l_i$. In this case,

$$k_i = 1 \quad \text{and} \quad q_i \mid -j_i q_i' A_{i,j_i} + b_{i,k_i} A_{i,j_i}\,,$$

which implies that $\operatorname{GCD}(-j_i q_i' + b_{i,k_i}, q_i) \neq 1$. Therefore, j_i must be a root of

$$R(z) = \operatorname{res}_x(b_{i,k_i}(x) - z q_i'(x), q_i(x)) \in C[z]\,.$$

First of all, $R(z) \neq 0$ for otherwise for some root β of $q(x)$, $b_{i,k_i} - z q'(\beta) = 0$ meaning $q_i'(\beta) = 0$ which contradicts the squarefreeness of q_i. Let m_i be the largest positive integral root of $R(z)$, if any, otherwise let $m_i = 0$. Then

$$j_i \le \bar{j}_i = \max(\min(l_i - 1, l_i - k_i)\,,\ \rho_i)\,.$$

We now set

$$y(x) = \frac{Y(x)}{q_1(x)^{\bar{j}_i} \cdots q_n(x)^{\bar{j}_n}} = \frac{Y(x)}{\bar{q}(x)}$$

and substitute into $(*)$. We get

$$uY' + vY = \sum_{i=1}^{m} c_i t_i \qquad (**)$$

with

$$\begin{aligned} Y(x) &= y_\alpha x^\alpha + \ldots + y_0 \in C[c_1,\ldots,c_m,x]\,, \\ u(x) &= a_\beta x^\beta + \ldots + a_0 \in C[x]\,, \\ v(x) &= b_\gamma x^\gamma + \ldots + b_0 \ \in C[x] \end{aligned}$$

and

$$\sum_{i=1}^{m} c_i t_i(x) = d_\delta x^\delta + \ldots + d_0 \in C[c_1,\ldots,c_m,x]$$

where all d_i are linear homogeneous elements of $C[c_1,\ldots,c_m]$. Again it behooves us to determine a bound for α. Substitution in $(**)$ gives

$$(a_\beta x^\beta + \ldots)(\alpha y_\alpha x^{\alpha-1} + \ldots) + (b_\gamma x^\gamma + \ldots)(y_\alpha x^\alpha + \ldots) = d_\delta x^\delta + \ldots \quad . \qquad (\dagger)$$

Thus $\alpha + \beta - 1 \le \delta$ if and only if $\alpha + \gamma \le \delta$ or the third case $\alpha + \beta - 1 = \alpha + \gamma > \delta$ which implies that $\alpha a_\beta + b_\gamma = 0$. Let ρ be $-b_\gamma / a_\beta$ if this is a positive integer, otherwise let $\rho = 0$. Then

$$\alpha \le \bar{\alpha} = \max(\min(\delta - \beta - 1, \delta - \gamma)\,,\ \rho)\,.$$

Taking $h_i = x^i/\bar{q}$, $0 \le i \le \bar{\alpha} = r$, multiplying (†) out and equating powers of x^i we obtain a linear system S in the y_i's and c_i's with coefficients in C. $\diamondsuit$

We now further inspect the third possibilities in our special case where f is the derivative of a rational function. The following theorem tells us that no unreasonably large bounds can occur.

Theorem 3.4: If one applies the algorithm given in the proof of theorem 3.3 to the differential equation $y' + f'y = g$, $f, g \in C[x]$, the third possibility for the bound $\bar{j}_i$ can never occur and the only time the third possibility for the bound $\bar{\alpha}$ can happen is when $\rho = \deg(\bar{q})$.

Proof: Assume that $j_i + 1 > l_i$ which implies that $k_i = 1$. Thus the partial fraction expansion

$$f'(x) = \frac{b_{i,k_i}(x)}{q_i(x)} + \ldots$$

which is impossible as shown in lemma 1.1. Now let

$$y(x) = \frac{Y(x)}{\bar{q}(x)}, \quad f'(x) = \frac{p(x)}{\hat{q}(x)}, \quad g(x) = \frac{s(x)}{q(x)}.$$

Notice that $\bar{q}(x)$ divides $q(x)$. Substituting into our differential equation $y' + f'y = g$ we get

$$\frac{Y'(x)}{\bar{q}(x)} + \left(\frac{p(x)}{\hat{q}(x)} - \frac{\bar{q}'(x)}{\bar{q}(x)}\right)\frac{Y(x)}{\bar{q}(x)} = \frac{s(x)}{q(x)} \tag{$*$}$$

Since the bound $\bar{\alpha}$ depends only on the difference $\delta - \beta$ and $\delta - \gamma$ as well as the quotient b_γ/a_β it does not matter for the determination of $\bar{\alpha}$ if we multiply $(*)$ with a larger than the least common denominator. We get

$$(\bar{q}\hat{q}q)Y' + q(p\bar{q} - \hat{q}\bar{q}')Y = \bar{q}^2\hat{q}s\,.$$

The third possibility implies that

$$\beta = \deg(\bar{q}\hat{q}q) = \gamma + 1 = \deg(q(p\bar{q} - \hat{q}\bar{q}')) + 1\,.$$

If $\deg(p) \ge \deg(\hat{q})$, this is clearly impossible. Thus $\deg(p) < \deg(\hat{q})$ which, since

$$\frac{p}{\hat{q}} = f' = \left(\frac{d}{e}\right)' = \frac{d'e - de'}{e^2}, \quad d, e \in C[x]\,,$$

implies that we may choose $\deg(d) < \deg(e)$ and thus get $\deg(p) \le \deg(\hat{q}) - 2$. Therefore, $a_\beta = \mathrm{ldcf}\,(\bar{q}\hat{q}q) = 1$, $b_\gamma = \mathrm{ldcf}\,(q(p\bar{q} - \hat{q}\bar{q}')) = \mathrm{ldcf}\,(-\bar{q}')$ and thus $-b_\gamma/a_\beta = \rho = \deg(\bar{q})$. $\diamondsuit$

We now present an example showing that the case

$$\deg(Y) = \deg(\bar{q}) > \max(0, \min(\delta - \beta - 1, \delta - \gamma))$$

can occur.

Example 3.1: Let $f' = -1/x^2$, $g = -(x+1)/x^4$. Then $q_1 = x$, $k_1 = 2$, $l_1 = 4$, $\bar{j_1} = \min(l_1 - 1, l_1 - k_1) = 2$ and

$$\left(\frac{Y}{x^2}\right)' - \frac{1}{x^2}\frac{Y}{x^2} = -\frac{x+1}{x^4} \quad \text{with} \quad \bar{q}(x) = x^2 .$$

This leads to

$$x^2 Y' - (2x+1)Y = -x - 1 .$$

Thus, $\beta = 2$, $\gamma = 1$, $\delta = 1$ and

$$\bar{\alpha} = \max(\min(\delta - \beta - 1, \delta - \gamma)\,,\ \deg(\bar{q})) = \max(\min(-2,-1), 2) = 2 .$$

Solving for $Y = y_2 x^2 + y_1 x + y_0$ we get $y_2 = 1$, $y_1 = -1$, $y_0 = 1$. Hence

$$\int \frac{x+1}{x^4} \exp\left(\frac{1}{x}\right) = -\frac{x^2 - x + 1}{x^2} \exp\left(\frac{1}{x}\right) .$$

Due to M. Rothstein [76], the degree bound $\bar{\alpha}$ for Y in $uY' + vY = t$ can be further reduced in the following way. If GCD $(u, v) \neq 1$ then we divide u, v and t by this GCD. Obviously, if the division of t leaves a remainder then the differential equation has no solution. Thus we may assume that GCD $(u, v) = 1$ and we can find unique polynomials $d, e \in C[x]$ with

$$ud + ve = t\,, \quad \deg(e) < \deg(u) .$$

Now $Y = \bar{Y}u + r$, $\deg(r) < \deg(u)$, if and only if

$$r = e \quad \text{and} \quad u\bar{Y}' + (u' + v)\bar{Y} = d - e' .$$

Thus solving for $\bar{Y}$ with $\deg(\bar{Y}) = \bar{\alpha} - \beta$ is sufficient. Of course, we can repeat this process until either $\deg(\bar{Y}) < \beta$ or $\deg(u) = 0$. In the first case $u\bar{Y}' + v\bar{Y} = t$ implies $\bar{Y} \equiv tv^{-1}$ (mod u). Thus we only need to invert v mod u. The second case must be handled by solving linear systems as discussed above.

Before proving theorem 3.1, we must establish another fairly deep fact.

Theorem 3.5: Let K be a differential field such that C_K is algebraically closed. Let $f_\alpha, g \in K[x_1, \ldots, x_n]$, $\alpha \in I$ with I a not necessarily finite set. Assume that $L \supset K$ and there exist $d_1, \ldots, d_n \in C_L$ such that $f_\alpha(d_1, \ldots, d_n) = 0$ for all $\alpha \in I$ and such that $g(d_1, \ldots, d_n) \neq 0$. Then there exist $c_1, \ldots, c_n \in C_K$ such that $f_\alpha(c_1, \ldots, c_n) = 0$ for all $\alpha \in I$ and $g(c_1, \ldots, c_n) \neq 0$.

Proof: Let $\{u_\beta\}_{\beta \in B}$ be a basis for the vector space K over C_K. We first prove that $\{u_\beta\}_{\beta \in B}$ remains linearly independent over C_L. Assume the contrary, that is there exist elements $u_0, u_1, \ldots, u_m \in \{u_\beta\}$ and $d_1, \ldots, d_m \in C_L$ such that

$$u_0 + d_1 u_1 + \ldots + d_m u_m = 0 .$$

Furthermore, assume that m is as small as possible. Since C_K was algebraically closed, at least one of the d_i, say d_1, must be transcendental over K. We now extend the derivation $'$ on

K to $\dot{}$ on $K(d_1, \ldots, d_m)$ such that $\dot{d}_1 = 1$ and $\dot{d}_i = 0$ for all $2 \le i \le m$ with d_i transcendental over $K(d_1, \ldots, d_{i-1})$. It is easy to see that $(\dot{d}_j)' = 0$ for all d_j algebraic over $K(d_1, \ldots, d_{j-1})$: We prove by induction on i that for any $a \in C_{K(d_1,\ldots,d_i)}$, $(\dot{a})' = 0$. For $i = 0$, $\dot{a} = a' = 0$.

First, assume that d_i is algebraic over $K(d_1, \ldots, d_{i-1})$. Let the minimal polynomial for d_i be

$$d_i^l + b_{l-1} d_i^{l-1} + \ldots + b_0 = 0\,, \quad b_i \in C_{K(d_1,\ldots,d_{i-1})}\,.$$

Then

$$\dot{d}_i = \frac{-\dot{b}_{l-1} d_i^{l-1} - \ldots - \dot{b}_0}{l d_i^{l-1} + \ldots + b_1}$$

and, since $d_i' = 0$, we obtain by induction hypothesis that $(\dot{d}_i)' = 0$. Similarly we can extend this fact to any $a \in K(d_1, \ldots, d_i)$ with $a' = 0$.

Secondly, let d_i be transcendental. Assume $a = f(d_i)/g(d_i)$ with $f, g \in K(d_1, \ldots, d_{i-1})[d_i]$ relatively prime, g monic, and $a' = 0$. Then

$$f(d_i)' g(d_i) - f(d_i) g(d_i)' = 0\,.$$

If $g(d_i)' \ne 0$, then $f(d_i)'/g(d_i)' = f(d_i)/g(d_i)$ which is a contradiction assuming that f/g is in reduced form. Hence $g(d_i)' = 0$ and thus $f(d_i)' = 0$, or otherwise d_i would be algebraic over $K(d_1, \ldots, d_{i-1})$. Therefore $f, g \in C_{K(d_1,\ldots,d_{i-1})}[d_i]$ and, since $(\dot{d}_i)' = 0$, we get $(\dot{a})' = 0$. This finishes the induction argument. Now the original claim follows quickly. For

$$\left(u_0 + \sum_{i=1}^m d_i u_i\right)' = u_0' + \sum_{i=1}^m d_i u_i' = 0$$

and

$$\begin{aligned}\left(u_0 + \sum_{i=1}^m d_i u_i\right)^{\cdot} &= \dot{u}_0 + \sum_{i=1}^m (d_i \dot{u}_i + \dot{d}_i u_i) \\ &= u_0' + \sum_{i=1}^m d_i u_i' + \sum_{i=1}^m \dot{d}_i u_i = u_1 + \sum_{i=2}^m \dot{d}_i u_i\,,\end{aligned}$$

because $\dot{}$ extends $'$. But this leads to a shorter linear combination of the u_i with coefficients in C_L ($(\dot{d}_i)' = 0$).

We now express $f_\alpha = \sum_{\beta \in B} u_\beta h_{\alpha\beta}$, $h_{\alpha\beta} \in C_K[x_1, \ldots, x_n]$, where for each $\alpha \in I$ only finitely many $h_{\alpha\beta}$ are non-zero. Since $\{u_\beta\}$ is linearly independent over C_L, $f_\alpha(d_1, \ldots, d_n) = 0$ implies $h_{\alpha\beta}(d_1, \ldots, d_n) = 0$ for all $\beta \in B$. We now consider the ideal

$$J = (h_{\alpha\beta})_{\alpha \in I, \beta \in B} \subseteq C_K[x_1, \ldots, x_n]\,.$$

Had $h_{\alpha\beta} = 0$ no solution in C_K, than $1 \in J$ (by the "weak" Hilbert Nullstellen-Satz stating that every nontrivial ideal in $F[x_1, \ldots, x_n]$, F an algebraically closed field, has a solution in its corresponding algebraic set $V(J)$), i.e. $1 = \sum A_{\alpha\beta} h_{\alpha\beta}$ with only finitely many $A_{\alpha\beta} \ne 0$. This contradicts the fact that

$$\sum A_{\alpha\beta}(d_1, \ldots, d_n) h_{\alpha\beta}(d_1, \ldots, d_n) = 0\,.$$

Finally, consider $g = \sum_{\beta\in B} u_\beta t_\beta$, $t_\beta \in C_K[x_1,\ldots,x_n]$. We observe that if for any $c_1,\ldots,c_n \in C_K$ one $t_\beta(c_1,\ldots,c_n) \neq 0$ then $g(c_1,\ldots,c_n) \neq 0$, since the u_β are algebraically independent. Therefore, assume that $t_\beta(c_1,\ldots,c_n) = 0$ for all $\beta \in B$ and all $(c_1,\ldots,c_n) \in V(J)$. Then by the Hilbert Nullstellen-Satz there exist integers r_β such that $t_\beta^{r_\beta} \in J$. Hence

$$t_\beta(d_1,\ldots,d_n)^{r_\beta} = \sum_{\gamma\in B} A_{\alpha\gamma}(d_1,\ldots,d_n)h_{\alpha\gamma}(d_1,\ldots,d_n) = 0\,,$$

meaning that $g(d_1,\ldots,d_n) = 0$, contradicting our original assumption. $\diamondsuit$

Proof of Theorem 3.1: Let

$$C_L K = \left\{ \frac{d_1k_1 + \ldots + d_nk_n}{d_{n+1}k_{n+1} + \ldots + d_mk_m} \;\middle|\; d_i \in C_L\,,\ k_i \in K\,,\ 1 \leq n < m \right\}.$$

Then $C_L K$ has the same field of constants as L and theorem 3.2 shows that there exist $v_0,\ldots,v_n \in C_L K$, $d_1,\ldots,d_n \in C_L$ such that

$$g' = v_0 + \sum_{i=1}^{n} d_i \frac{v_i'}{v_i}\,.$$

In fact, there exist constants $d_1,\ldots,d_m \in C_L$, $m \geq n$, such that $v_0,\ldots,v_n \in K(d_1,\ldots,d_m)$. As in the proof of theorem 3.2, we can assume that $v_1,\ldots,v_n \in K[d_1,\ldots,d_m]$ and $v_0 = p/q$, $p,q \in K[d_1,\ldots,d_m]$. Multiplying the denominators out in the above equation we get

$$v_1\cdots v_n(g'q^2 - p'q + pq') - \sum_{i=1}^{n} d_i q^2 \left(v_i' \prod_{j\neq i} v_j \right) = 0$$

and

$$q(d_1,\ldots,d_m) \prod_{i=1}^{n} v_i(d_1,\ldots,d_m) \neq 0\,.$$

Applying theorem 3.5 to these equations we find constants $c_1,\ldots,c_m \in \bar{C}K$ such that

$$q(c_1,\ldots,c_m) \prod_{i=1}^{n} v_i(c_1.\ldots,c_m) \neq 0$$

and

$$g' = \frac{p(c_1,\ldots,c_m)}{q(c_1,\ldots,c_m)} + \sum_{i=1}^{n} c_i \frac{v_i(c_1,\ldots,c_m)'}{v_i(c_1,\ldots,c_m)}\,.$$

Taking the trace in the normal closure of $K(c_1,\ldots,c_m)$ over K we obtain the first summand to be an element in K. $\diamondsuit$

Historic Remarks: Inserted Later.

4. Integration in Regular Elementary Liouville Extensions

We now assume that an element $f \in C(x, \theta_1, \ldots, \theta_n)$, $C \subset \mathbb{C}$ finitely generated over $\mathbb{Q}$, $C(x)$ the rational functions over C (i.e. $x' = 1$), θ_i a monomial over $C(x, \theta_1, \ldots, \theta_{i-1})$, is given to us together with an algorithm capable of performing arithmetic in C. Then we want to find an elementary integral of f,

$$\int f = v_0 + \sum_{i=1}^{n} c_i \log v_i \,,$$

as established in theorem 3.1, provided it exists. Otherwise, we wish to positively verify that no such integral can be found. We will perceive $f \in K(\theta)$, $\theta = \theta_n$, $K = C(x, \theta_1, \ldots, \theta_{n-1})$ and start out as in the rational case. The following theorem shows how to peel off part of v_0 and simplify the problem.

Theorem 4.1: Let $K(\theta) \supset K$ be differential fields, θ a monomial over K, $p(\theta), q(\theta) \in K[\theta]$, q monic, $\mathrm{GCD}\,(p, q) = 1$.

a) $\theta = \log \eta$, $\eta \in K$: Let $\bar{q} = \mathrm{GCD}\,(q, dq/d\theta)$ and $q^* = q/\bar{q}$. Then there exist unique polynomials $f, g, h \in K[\theta]$ such that

$$\frac{p(\theta)}{q(\theta)} = f(\theta) + \left(\frac{g(\theta)}{\bar{q}(\theta)}\right)' + \frac{h(\theta)}{q^*(\theta)} \,,$$

where $\deg(g) < \deg(\bar{q})$, $\deg(h) < \deg(q^*)$ and $\deg(f) = \deg(p) - \deg(q)$ if the latter is ≥ 0, otherwise $f = 0$.

b) $\theta = \exp \eta$, $\eta \in K$. Let $q = \hat{q}\theta^k$ such that $\theta \not| \; \hat{q}$ and let $\bar{q} = \mathrm{GCD}\,(\hat{q}, d\hat{q}/d\theta)$ and $q^* = \hat{q}/\bar{q}$. Then there exist $f^+, f^-, g, h \in K[\theta]$ such that

$$\frac{p(\theta)}{q(\theta)} = f^+(\theta) + \frac{f^-(\theta)}{\theta^k} + \left(\frac{g(\theta)}{\bar{q}(\theta)}\right)' + \frac{h(\theta)}{q^*(\theta)} \,,$$

where $\deg(g) < \deg(\bar{q})$, $\deg(h) < \deg(q^*)$, $\deg(f^-) < k$, $\deg(f^+) = \deg(p) - \deg(q)$ if the latter is ≥ 0, otherwise $f^+ = 0$.

Proof: If $\theta = \log \eta$ then we set $\hat{q} = q$. Now let $t_1 t_2^2 \cdots t_r^r$ be the squarefree decomposition of $\hat{q}$. We can express $q^* = t_1 \cdots t_r$ and $\bar{q} = t_2 \cdots t_r^{r-1}$. Furthermore, if g and h exist then

$$\left(\frac{g}{\bar{q}}\right)' + \frac{h}{q^*} = \frac{b}{\hat{q}} \quad \text{with} \quad b = g' q^* - g \sum_{i=2}^{r} (i-1) \frac{q^* t_i'}{t_i} + \bar{q} h \,,$$

hence $\deg_\theta(b) < \deg_\theta(\hat{q})$. Therefore, the polynomials f, f^+, f^- and b are the unique polynomials obtained by the partial fraction decomposition

$$\frac{p}{\hat{q}} = f + \frac{b}{\hat{q}} \qquad \text{if} \quad \theta = \log \eta$$

and

$$\frac{p}{\theta^k \hat{q}} = f^+ + \frac{f^-}{\theta^k} + \frac{b}{\hat{q}} \qquad \text{if} \quad \theta = \exp \eta \,.$$

It remains to show that $b/\hat{q}$ can be uniquely decomposed into $(g/\hat{q})' + h/q^*$. We prove this by induction on r, the highest multiplicity of a squarefree factor of $\hat{q}$.

For $r = 1$ the only solution to $b/\hat{q} = (g/1)' + h/\hat{q}$ is $g = 0$ and $h = b$, since $\deg(g) < 0$.

Induction Argument: Let $u, v \in K[\theta]$ be the polynomials solving

$$-u\frac{q^*\bar{q}'}{\bar{q}} + v\bar{q} = b\,, \quad \deg(u) < \deg(\bar{q})\,, \quad \deg(v) < \deg(q^*)\,.$$

Since $q^*\bar{q}'/\bar{q} = \sum_{i=2}^{r}(i-1)q^*t_i'/t_i$ is relatively prime to $\bar{q}$, such polynomials u, v exist and are uniquely determined. Therefore

$$\frac{b}{\hat{q}} = \left(\frac{u}{\bar{q}}\right)' + \frac{v}{q^*} - \frac{u'}{\bar{q}}$$

But $\bar{q} = t_2 \cdots t_r^{r-1}$ and we can apply the induction hypothesis to $-u'/\bar{q}$ and determine unique polynomials $\tilde{u}$, $\tilde{v}$ such that

$$\frac{-u'}{\bar{q}} = \left(\frac{\tilde{u}}{t_3 \cdots t_r^{r-2}}\right)' + \frac{\tilde{v}}{t_2 \cdots t_r}\,, \quad \deg(\tilde{u}) < \deg(t_3 \cdots t_r^{r-2})\,,$$
$$\deg(\tilde{v}) < \deg(t_2 \cdots t_r)\,.$$

Therefore

$$\frac{b}{\hat{q}} = \left(\frac{u + t_2 \cdots t_r \tilde{u}}{\bar{q}}\right)' + \frac{v + t_1\tilde{v}}{q^*}$$

and the polynomials $g = u + t_2 \cdots t_r \tilde{u}$, $h = v + t_1\tilde{v}$ satisfy also the degree constraints. The uniqueness follows as in lemma 1.1, using lemma 3.2. $\diamondsuit$

In general, the sum of two non-elementary integrals can become elementary, e.g. $\int \exp(x^2)\,dx$ and $\int 1 - \exp(x^2)\,dx$. In our case, however $\int p/q$ is elementary if and only if

$$\int f(\theta) \quad \text{or} \quad \int f^+(\theta) \text{ and } \int \frac{f^-(\theta)}{\theta^k}\,, \quad \text{and} \quad \int \frac{h(\theta)}{q^*(\theta)}$$

are elementary. The reason is that, by the Liouville theorem, we know the form of an answer. Then comparing the partial fraction expansions of the integrand and the derivation of such an answer shows that individual parts match separately. We first investigate the conditions under which $\int h(\theta)/q^*(\theta)$ is elementary.

Theorem 4.2: Let $K(\theta) \supset K$ be differential fields, θ a monomial over K, $p(\theta), q(\theta) \in K[\theta]$ such that $q(\theta) = \theta^m + b_{m-1}\theta^{m-1} + \ldots + b_0$ is squarefree, $m \geq 1$, $\deg(p) < \deg(q)$ and $\mathrm{GCD}\,(p, q) = 1$. In case $\theta = \exp\eta, \eta \in K$ we moreover assume that $b_0 \neq 0$. Then $\int p(\theta)/q(\theta)$ is elementary if and only if all roots $\zeta_1, \ldots, \zeta_l$ of

$$R(z) = \mathrm{res}_\theta(p(\theta) - zq'(\theta)\,,\ q(\theta)) \in K[z]$$

are constants in $\bar{C}_K$. Furthermore, in this case

a) If $\theta = \log \eta$, $\eta \in K$, then

$$\int \frac{p(\theta)}{q(\theta)} = \sum_{i=1}^{l} \zeta_i \log v_i(\theta)$$

with $v_i(\theta) = \mathrm{GCD}\,(p(\theta) - \zeta_i q'(\theta), q(\theta)) \in \bar{C}_K K[\theta]$ such that v_i is monic.

b) If $\theta = \exp \eta$, $\eta \in K$, then

$$\int \frac{p(\theta)}{q(\theta)} = \sum_{i=1}^{l} (-n_i \zeta_i \eta + \zeta_i \log v_i(\theta))$$

where $v_i(\theta) = \mathrm{GCD}\,(p(\theta) - \zeta_i q'(\theta),\ q(\theta)) \in \bar{C}_K K[\theta]$ such that v_i is monic, and $n_i = \deg_\theta(v_i)$.

Proof: Let $\alpha_1, \ldots, \alpha_m \in \bar{K}$ be the roots of $q(\theta)$, thus $\alpha_i \neq \alpha_j$ for $i \neq j$ since q squarefree. We now can compute the partial fraction decomposition of p/q,

$$\frac{p(\theta)}{q(\theta)} = \frac{k_1}{\theta - \alpha_1} + \ldots + \frac{k_m}{\theta - \alpha_m}\,, \quad k_i \in \bar{K}\,, \quad 1 \leq i \leq m\,.$$

Let

$$t_i = \begin{cases} \dfrac{k_i}{\eta'/\eta - \alpha_i'} & \text{if} \quad \theta = \log \eta\,,\ \eta \in K \\[2ex] \dfrac{k_i}{\eta' \alpha_i - \alpha_i'} & \text{if} \quad \theta = \exp \eta\,,\ \eta \in K \end{cases}$$

By lemma 2.4 we conclude that neither $\eta'/\eta = \alpha_i'$, if $\theta = \log \eta$, nor $\eta' = \alpha_i'/\alpha_i$, if $\theta = \exp \eta$ (by assumption all $\alpha_i \neq 0$). Now

$$t_i \frac{(\theta - \alpha_i)'}{\theta - \alpha_i} = \frac{k_i}{\theta - \alpha_i} \qquad \text{if} \quad \theta = \log \eta$$

and

$$-t_i \eta' + t_i \frac{(\theta - \alpha_i)'}{\theta - \alpha_i} = \frac{k_i}{\theta - \alpha_i} \qquad \text{if} \quad \theta = \exp \eta\,.$$

Therefore

$$p(\theta) = \begin{cases} \sum\limits_{i=1}^{l} t_i (\theta - \alpha_i)' \prod\limits_{j \neq i} (\theta - \alpha_j) & \text{if} \quad \theta = \log \eta \\ \sum\limits_{i=1}^{l} (-t_i \eta' (\theta - \alpha_i) + t_i (\theta - \alpha_i)') \prod\limits_{j \neq i} (\theta - \alpha_j) & \text{if} \quad \theta = \exp \eta \end{cases}$$

and thus $p(\alpha_i) = t_i q(\alpha_i)'$ or $t_i = p(\alpha_i)/q(\alpha_i)'$.

If: Since $p(\alpha_i) - t_i q'(\alpha_i) = 0$, $\theta - \alpha_i \mid p(\theta) - t_i q'(\theta)$. Therefore $\mathrm{GCD}\,(p(\theta) - t_i q'(\theta),\ q(\theta)) \neq 0$ or $R(t_i) = 0$. Hence t_i is a constant for all $1 \leq i \leq m$ and

$$\int \frac{p(\theta)}{q(\theta)} = \begin{cases} \sum\limits_{i=1}^{m} t_i \log(\theta - \alpha_i) & \text{if} \quad \theta = \log \eta \\ \sum\limits_{i=1}^{m} (-t_i \eta + t_i \log(\theta - \alpha_i)) & \text{if} \quad \theta = \exp \eta \end{cases}$$

is apparently elementary.

Only If: By the strong Liouville theorem

$$\int \frac{p(\theta)}{q(\theta)} = v_0(\theta) + \sum_{i=1}^{n} c_i \log v_i(\theta)$$

with $v_0(\theta) \in K(\theta)$, $c_i \in \bar{C}_K$, $v_i(\theta) \in \bar{C}_K K[\theta]$ monic, squarefree and pairwise relatively prime. Then, arguing as in the proof of theorem 3.2, we conclude that

$$v_0(\theta) = \begin{cases} 0\,, & \text{if} \quad \theta = \log \eta \\ -\sum_{i=1}^{n} c_i n_i \eta\,, \quad \text{with } n_i = \deg v_i\,, & \text{if} \quad \theta = \exp \eta \end{cases}$$

Let β_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n_i$, be the roots of v_i in $\bar{K}$. Then

$$\frac{p(\theta)}{q(\theta)} = \begin{cases} \sum_{i,j} c_i \dfrac{(\theta - \beta_{ij})'}{\theta - \beta_{ij}} & \text{if} \quad \theta = \log \eta \\ \sum_{i,j} -c_i \eta' + c_i \dfrac{(\theta - \beta_{ij})'}{\theta - \beta_{ij}} & \text{if} \quad \theta = \exp \eta \end{cases}$$

which, by the uniqueness of the partial fraction decomposition, implies that for each t_i there exists a c_{j_i}, $1 \leq j_i \leq n$ with $t_i = c_{j_i}$.

Now let γ be a root of $R(z)$. Then GCD $(p(\theta) - \gamma q'(\theta), q(\theta)) \neq 1$, hence there exists an α_i such that $\theta - \alpha_i \mid p(\theta) - \gamma q'(\theta)$. This means that $p(\alpha_i) - \gamma q'(\alpha_i) = 0$ or that $\gamma = p(\alpha_i)/q'(\alpha_i) = t_i = c_{j_i}$, which is a constant.

Extensions a) and b) now follow easily. For, as just shown, each root γ of $R(z)$ occurs among the t_i and vice-versa. Furthermore,

$$\mathrm{GCD}\,(p(\theta) - t_i q'(\theta), q(\theta)) = \prod_{j:t_j = t_i} (\theta - \alpha_j) = v_i(\theta)$$

hence

$$\sum_{j:t_j = t_i} t_j \frac{(\theta - \alpha_j)'}{\theta - \alpha_j} = t_i \frac{v_i(\theta)'}{v_i(\theta)}\,. \quad \diamondsuit$$

Notice that if K is a regular elementary purely transcendental Liouville extension of $C(x)$ then it is easy to verify that a polynomial $R(z) \in K[z]$ has only constant roots. For the minimal polynomial of constant roots have constant coefficients and therefore

$R(z) = r_j z^j + \ldots + r_0$ has all constant roots
if and only if $\dfrac{r_i}{r_j} \in C$ for all $1 \leq i \leq j$.

The latter condition can be verified by polynomial division in $C[x, \theta_1, \ldots, \theta_n]$.

We now investigate under which conditions $\int f(\theta)$ is elementary, provided that $\theta = \log \eta$. Again by the Liouville theorem there must exist $b_{m+1} \in C_K$, $b_m, \ldots, b_0 \in K$, $c_1, \ldots, c_j \in \bar{C}_K$, $w_1, \ldots, w_j \in \bar{C}_K K$ such that

$$\int f(\theta) = \int (a_m \theta^m + \ldots + a_0) = b_{m+1}\theta^{m+1} + \ldots + b_0 + \sum_{i=1}^{j} c_j \log w_j \,.$$

For this to hold we must have

$$\int a_m = b_m + (m+1) b_{m+1} \log \eta \,, \quad b_m = \bar{b}_m + \hat{b}_m \,, \quad \hat{b}_m \in C_K \,,$$

$$\int a_{m-1} - m \bar{b}_m \frac{\eta'}{\eta} = b_{m-1} + m \hat{b}_m \log \eta \,, \quad bm - 1 = \bar{b}_{m-1} + \hat{b}_{m-1} \,, \quad \hat{b}_{m-1} \in C_K \,,$$

$$\vdots$$

$$\int a_0 - \bar{b}_1 \frac{\eta'}{\eta} = b_0 + \hat{b}_1 \log \eta + \sum_{i=1}^{j} c_j \log w_j \,.$$

Therefore all integrals on the left-hand sides must be elementary. We may assume (by induction hypothesis or recursion) that for any element $a \in K = C(x, \theta_1, \ldots, \theta_{n-1})$ we can compute $v_0 \in K$, $c_1, \ldots, c_l \in \bar{C}$, $v_1, \ldots, v_l \in \bar{C}[x, \theta_1, \ldots, \theta_{n-1}]$ squarefree, pairwise relatively prime such that

$$\int a = v_0 + \sum_{i=1}^{l} c_i \log v_i \,,$$

provided that this integral is elementary. Doing so for the above integrals, $\bar{b}_i$, $1 \le i \le m$, $\hat{b}_i$, $2 \le i \le m$, exist if and only if the squarefree factors of the numerator and the denominator of η can be matched with the v_i and the c_i are integral multiples of one constant with multiplicities corresonding to the multiplicities of the squarefree factors. Notice that in the last case it is necessary and sufficient that $\int a_0 - \bar{b}_1 \eta'/\eta$ is elementary.

Finally, we investigate under which conditions $\int f^+(\theta)$, $\int f^-(\theta)/\theta^k$ are elementary for $\theta = \exp \eta$. There must exist $b_m, \ldots, b_1$, $b_{-1}, \ldots, b_{-k} \in K$ such that

$$\int f^+(\theta) = \int (a_m \theta^m + \ldots + a_0) = b_m \theta^m + \ldots + b_1 \theta + \int a_0$$

and $\int a_0$ must be elementary, as well as

$$\int f^-(\theta) = \int (a_{-1}\theta_{-1} + \ldots + a_{-k}\theta^{-k}) = b_{-1}\theta^{-1} + \ldots + b_{-k}\theta^{-k} \,.$$

Therefore we need to construct solutions $b_m, \ldots, b_{-k} \in K$. As the differential equations

$$\begin{array}{rclcrcl} b'_m &=& k\eta' b_m + a_k \,, & \qquad & b'_{-k} - k\eta' b_{-k} &=& a_{-k} \\ & \vdots & & & & \vdots & \\ b'_1 &=& \eta' b_1 + a_1 \,, & & b'_{-1} - \eta' b_{-1} &=& a_{-1} \,. \end{array}$$

In general, we want to solve $y' + f'y = g$, $f, g \in K$ for $y \in K$. Notice that for $f = \pm i\eta$ the solution is unique since otherwise $(y_1 - y_2)'/(y_1 - y_2) = \mp i\eta'$ contradicting that θ is a mononial (cf. lemma 2.4). Notice further that for $K = C(x)$ we already gave an algorithm. In fact, we shall prove a copy of theorem 3.3, replacing C by $K = C(x, \theta_1, \ldots, \theta_{n-1})$ and x by θ. Our assumption is that we can compute elementary integrals in K and solve similar differential equations over K.

Theorem 4.3: Let $K(\theta) \supset K$ be regular elementary purely transcendental Liouville extensions of $C(x)$, $C = C_K$. Assume that $f, g_1, \ldots, g_m \in K(\theta)$ are given. Consider the differential equation

$$y' + fy = \sum_{i=1}^{m} c_i g_i \quad \text{with} \quad c_1, \ldots, c_n \in C \tag{$*$}$$

in $y \in K(x)$. Then we can compute, in a finite number of arithmetic operations in C (including computation of integral roots of polynomials in $C[z]$), elements $h_1, \ldots, h_r \in K(x)$ and a linear system S with coefficients in C and variables $z_1, \ldots, z_r$, $e_1, \ldots, e_m$ such that

y solves $(*)$ if and only if

$$y = \sum_{i=1}^{r} y_i h_i \quad \text{and} \quad z_1 = y_1, \ldots, z_r = y_r\,, \quad e_1 = c_1, \ldots, e_m = c_m \quad \text{solve} \quad S\,.$$

Proof: We perform induction on the number of monomials of $K(\theta)$. By theorem 3.3 the statement is true if we have no monomial at all. If the statement is true for $K(\theta) = C(x, \theta_1, \ldots, \theta_{n-1})$ then by the previous elaboration we can find elementary integrals in $C(x, \theta_1, \ldots, \theta_n)$. We will also use this fact in addition to the induction hypothesis. The proof now proceeds as the one for theorem 3.3. Here is the outline: Let

$$y(\theta) = \frac{Y(\theta)}{q_1(\theta)^{j_1} \cdots q_\nu(\theta)^{j_\nu}}$$

with $Y(\theta) \in K[c_1, \ldots, c_m, \theta]$, $q_i(\theta) \in K[\theta]$ the monic, squarefree, pairwise relatively prime factors of the denominators of $f(\theta)$ and $g_j(\theta)$.

a) Find a bound $\bar{j}_i \geq j_i$ for all $1 \leq i \leq \nu$.

b) Using these bounds, compute a bound $\bar{\alpha} \geq \deg_\theta(Y)$.

c) Find the $\bar{\alpha} + 1$ coefficients of $Y(\theta)$ in $K[c_1, \ldots, c_m]$.

We now present each step for the cases $\theta = \log \eta$ and $\theta = \exp \eta$.

a) *Case* $\theta = \log \eta$: The bound is computed exactly as it was in theorem 3.3. In order to find integer roots of $R(z) \in K[z]$ we rewrite $R(z)$ to be an element in $(C[z])(x, \theta_1, \ldots, \theta_{n-1})$ and compute the GCD of the coefficients of the numerator being elements in $C[z]$. A constant root must then be a root of this GCD.

Case $\theta = \exp \eta$: If $q_i \neq \theta$ the bound is the same as in the logarithmic case. Notice that lemma 3.2 applies again. Now let $q_i = \theta$. Then for

$$y(x) = \frac{A_{i,j_i}(\theta)}{\theta^{j_i}} + \cdots, \quad f(x) = \frac{b_{i,k_i}(\theta)}{\theta^{k_i}} + \cdots, \quad \sum_{s=1}^{m} c_s g_s(\theta) = \frac{B_{i,l_i}(\theta)}{\theta^{l_i}} + \cdots,$$

being the partial fraction decompositions of y, f and $\sum c_s g_s$, $A_{i,j_i}, B_{i,l_i} \in K[c_1, \ldots, c_m]$, $b_{i,k_i} \in K$, the differential equation $(*)$ is

$$\frac{A'_{i,j_i} - j_i A_{i,j_i}\eta'}{\theta^{j_i}} + \ldots + \frac{b_{i,k_i} A_{i,j_i}}{\theta^{j_i+k_i}} + \ldots + \frac{B_{i,l_i}}{\theta^{l_i}} + \ldots \quad .$$

Then $j_i \le l_i$ is equivalent to $j_i + k_i \le l_i$. The third possibility is $j_i = j_i + k_i > l_i$ which implies that

$$\frac{A'_{i,j_i}}{A_{i,j_i}} - j_i\eta' + b_{i,k_i} = 0\,.$$

Therefore j_i is bounded by the constant of

$$\int b_{i,k_i} = j_i\eta - \log A_{i,j_i}\,.$$

To find j_i, we must integrate b_{i,k_i} and check whether the rational part is of the form a positive integer times η. If this is the case then we must take $\bar{j}_i = \max(l_i - k_i,$ *this integer coefficient*$)$.

b) We expand $(*)$ for $y(\theta) = Y(\theta)/\bar{q}(\theta)$ as in theorem 3.3. We get

$$uY' + vY = \sum_{i=1}^{m} c_i t_i \qquad (**)$$

with

$$\begin{aligned} Y(\theta) &= y_\alpha\theta^\alpha + \ldots + y_0 \in K[c_1, \ldots, c_m, \theta]\,, \\ u(\theta) &= a_\beta\theta^\beta + \ldots + a_0 \in K[\theta]\,, \quad a_\beta \ne 0\,, \\ v(\theta) &= b_\gamma\theta^\gamma + \ldots + b_0 \in K[\theta]\,, \quad b_\gamma \ne 0 \end{aligned}$$

and

$$\sum_{i=1}^{m} c_i t_i(\theta) = d_\delta\theta^\delta + \ldots + d_0 \in K[c_1, \ldots, c_m, \theta]$$

where all d_i are linear homogeneous elements of $K[c_1, \ldots, c_m]$.

Case $\theta = \log\eta$: Substituting in $(**)$ gives

$$\begin{aligned} &(a_\beta\theta^\beta + \ldots)(y'_\alpha\theta^\alpha + (y'_{\alpha-1} + \alpha\frac{\eta'}{\eta}y_\alpha)\theta^{\alpha-1} + \ldots) \\ &\qquad + (b_\gamma\theta^\gamma + \ldots)(y_\alpha\theta^\alpha + \ldots) = d_\delta\theta^\delta + \ldots \quad . \qquad (\dagger) \end{aligned}$$

Subcase $y'_\alpha = 0$: Then $\alpha + \beta - 1 \le \delta$ is equivalent to $\alpha + \gamma \le \delta$ or, as the third possibility, $\alpha + \beta - 1 = \alpha + \gamma > \delta$. Then

$$a_\beta\left(y'_{\alpha-1} + \alpha\frac{\eta'}{\eta}y_\alpha\right) + b_\gamma y_\alpha = 0$$

or, using the fact that y_α is a constant,

$$\int \frac{b_\gamma}{a_\beta} = -\frac{y_{\alpha-1}}{y_\alpha} - \alpha\log\eta\,.$$

Integrating b_γ/a_β we thus can find a bound for this exceptional case.

Subcase $y'_\alpha \neq 0$: In this case we either use $\alpha+\beta \leq \delta+1$ which is equivalent to $\alpha+\gamma \leq \delta+1$ or, as the third possibility, $\alpha+\beta = \alpha+\gamma > \delta+1$. Then both $a_\beta y'_\alpha + b_\gamma y_\alpha = 0$ and

$$a_\beta y'_{\alpha-1} + b_\gamma y_{\alpha-1} + a_{\beta-1} y'_\alpha + \left(\alpha\frac{\eta'}{\eta}a_\beta + b_{\gamma-1}\right) y_\alpha = 0\,.$$

Letting $y_{\alpha-1} = v y_\alpha$, $v \in K$, we have

$$a_\beta y_\alpha v' + (a_\beta y'_\alpha + b_\gamma y_\alpha)v + a_{\beta-1}y'_\alpha + (b_{\gamma-1} + \alpha\frac{\eta'}{\eta}a_\beta)y_\alpha = 0\,.$$

Now dividing by $a_\beta y_\alpha$ and using $y'_\alpha/y_\alpha = -b_\gamma/a_\beta$,

$$v' - \frac{a_{\beta-1}b_\gamma}{a_\beta^2} + \frac{b_{\gamma-1}}{a_\beta} + \frac{\alpha\eta'}{\eta} = 0$$

thus

$$\int \frac{a_{\beta-1}b_\gamma - a_\beta b_{\gamma-1}}{a_\beta^2} = v + \alpha\log\eta\,.$$

Therefore, in this exceptional case, we can bound α be the constant obtained for the above integral.

We set $\bar{\alpha}$ to the maximum of $\min(\delta+1-\beta, \delta+1-\gamma)$ and possible positive integers found in both third possibilities.

Case $\theta = \exp\eta$: Substituting in $(**)$ gives

$$(a_\beta\theta^\beta + \ldots)((y'_\alpha + \alpha\eta' y_\alpha)\theta^\alpha + \ldots) + (b_\gamma\theta^\gamma + \ldots)(y_\alpha\theta^\alpha + \ldots) = d_\delta\theta^\delta + \ldots \quad . \tag{‡}$$

Either $\alpha+\beta \leq \delta$ which is equivalent to $\alpha+\gamma \leq \delta$ or, as the third possibility, $\alpha+\beta = \alpha+\gamma > \delta$. In this case $a_\beta y'_\alpha + (\alpha\eta' a_\beta + b_\gamma)y_\alpha = 0$. Then

$$\int \frac{b_\gamma}{a_\beta} = -\alpha\eta - \log y_\alpha$$

and we can again find whether such an integer exists. Let $\bar{\alpha}$ be the maximum of $\min(\delta-\beta, \delta-\gamma)$ and the integer found in the third case.

c) We expand (†) to

$$\begin{aligned}(a_\lambda\theta^\lambda + \ldots)(y'_{\bar{\alpha}}\theta^{\bar{\alpha}} + (y'_{\bar{\alpha}-1} + \bar{\alpha}\frac{\eta'}{\eta}y_{\bar{\alpha}})\theta^{\bar{\alpha}-1} + \ldots)& \\ + (b_\lambda\theta^\lambda + \ldots)(y_{\bar{\alpha}}\theta^{\bar{\alpha}} + \ldots) &= d_{\lambda+\bar{\alpha}}\theta^{\lambda+\bar{\alpha}} + \ldots\end{aligned} \tag{§}$$

by adding zero coefficients in front of $u = a_\beta\theta^\beta + \ldots$ and $v = b_\gamma\theta^\gamma + \ldots$. We can make a_λ or b_λ non-zero by, if this is not the case, imposing linear relations S_1 on $c_1, \ldots, c_m$ resulting from the condition $d_{\lambda+\bar{\alpha}} = 0$. We now construct by induction on $\bar{\alpha}$ a linear system S in

$m+s+t$ variables over C and polynomials $h_1, \dots, h_{s+t} \in K[\theta]$, all of degree not more than $\bar{\alpha}$, such that

$$Y = y_{\bar{\alpha}} \theta^{\bar{\alpha}} + \dots \quad \text{satisfies} \quad (\S)$$

$$\text{if and only if} \quad Y = \sum_{i=1}^{s+t} y_i h_i \quad \text{and} \quad y_1, \dots, y_{s+t}\,,\ c_1, \dots, c_m \quad \text{solve } S\,.$$

$\bar{\alpha} = 0$: Then $(\S)$ takes the form

$$(a_\lambda \theta^\lambda + \dots) y_0' + (s_\lambda \theta^\lambda + \dots) y_0 = d_\lambda \theta^\lambda + \dots \quad . \tag{@}$$

Looking at the leading coefficient of θ_λ, we get $a_\lambda y_0' + s_\lambda y_0 = d_\lambda$ which, by induction hypothesis of our theorem, is equivalent to $y_0 = \sum_{i=1}^{s} e_i h_i$, $h_i \in K$ and $e_1, \dots, e_s$, $c_1, \dots, c_m$ solve a linear system S_2 over C. Substituting this expression for y_0 in (@) we get further linear equations, first over K, but then using the fact that K is transcendentally generated over C, a linear system S_3 in $e_1, \dots, e_s$, $c_1, \dots, c_m$ over C. Thus $S = S_1 \cup S_2 \cup S_3$.

Assume now that our statement is true for $\bar{\alpha} - 1$. Now the coefficient of $\theta^{\lambda + \bar{\alpha}}$ in $(\S)$ is

$$a_\lambda y_{\bar{\alpha}}' + b_\lambda y_{\bar{\alpha}} = d_{\lambda + \bar{\alpha}}$$

resulting, as above, in $h_1, \dots, h_s \in K$ and a linear system S_2 in $e_1, \dots, e_s$, $c_1, \dots, c_m$ such that $y_{\bar{\alpha}}$ solves the above differential equation if and only if $y_{\bar{\alpha}} = \sum_{i=1}^{s} e_i h_i$ and $e_1, \dots, e_s$, $c_1, \dots, c_m$ solve S_2. Substituting this expression for $y_{\bar{\alpha}}$ in $(\S)$ gives

$$(a_\lambda \theta^\lambda + \dots)(y_{\bar{\alpha}-1}' \theta^{\bar{\alpha}-1} + \dots) + (b_\lambda \theta^\lambda + \dots)(y_{\bar{\alpha}-1} \theta^{\bar{\alpha}-1} + \dots)$$

$$= d_{\lambda + \bar{\alpha} - 1} \theta^{\lambda + \bar{\alpha} - 1} + \dots - (a_{\lambda-1} \theta^{\lambda-1} + \dots) y_{\bar{\alpha}}' \theta^{\bar{\alpha}} - (a_\lambda \theta^\lambda + \dots) \bar{\alpha} \frac{\eta'}{\eta} y_{\bar{\alpha}} \theta^{\bar{\alpha}-1} - (b_{\lambda-1} \theta^{\lambda-1} + \dots) y_{\bar{\alpha}} \theta^{\bar{\alpha}}\,.$$

To this equation we can apply our induction hypothesis and obtain $h_{s+1}, \dots, h_{s+t} \in K[\theta]$, with degrees no more than $\bar{\alpha}-1$, and a linear system S_3 in $e_1, \dots, e_s$, $e_{s+1}, \dots, e_{s+t}$, $c_1, \dots, c_m$ over C such that

$$Y = y_{\bar{\alpha}-1} \theta^{\bar{\alpha}-1} + \dots \quad \text{solves the above equation if and only if} \quad Y = \sum_{i=s+1}^{s+t} e_i h_i$$

$$\text{and the} \quad e_{s+1}, \dots, e_{s+t} \quad \text{solve } S_3\,.$$

We now obtain $S = S_1 \cup S_2 \cup S_3$ and the h's as $h_1 \theta^{\bar{\alpha}}, \dots, h_s \theta^{\bar{\alpha}}$, $h_{s+1}, \dots, h_{s+t}$.

The process for solving (‡) is exactly the same. This concludes the proof of theorem 4.3. $\diamondsuit$

Appendix

For the proof of theorem 2.2 we introduce, following M. Rosenlicht [76], the notion of *free module of C-differentials* of a field K. Let $K \supseteq C$ be fields, let Φ be the free K-module generated by the symbols $\{\delta x\}_{x \in K}$ and let Ψ be its submodule generated by

$$\{\delta(x+y) - \delta x - \delta y\,,\ \delta(xy) - x\delta y - y\delta x\}_{x,y \in K} \cup \{\delta c\}_{c \in C}\,.$$

Then $\Omega_{K/C} = \Phi/\Psi$ denotes the free module of C-differentials of K. Let $d\colon K \to \Omega_{K/C}$ be defined as $d(x) = \delta x \bmod \Psi$, then for any derivation $D\colon K \to K$ with constants C there exists a unique K-homomorphism $t\colon \Omega_{K/C} \to K$, K interpreted as a K-module, such that $D = td$. If we define D as a C-derivation from K into a K-module M with $D(x+y) = Dx + Dy$, $D(xy) = xDy + yDx$, $D(cx) = cDx$ for all $x, y \in K$, $c \in C$, then $\Omega_{K/C}$ is the free structure of such maps.

Theorem 2.3: Let $C \subseteq K$ be fields (of characteristic 0), let $u_1, \ldots, u_n$, $v \in K$ with $u_1 \neq 0, \ldots, u_n \neq 0$, and let $c_1, \ldots, c_n \in C$ be linearly independent over $\mathbb{Q}$. Then

$$c_1 \frac{du_1}{u_1} + \ldots + c_n \frac{du_n}{u_n} + dv \in \Omega_{K/C}$$

is zero if and only if each $u_1, \ldots, u_n$, v is algebraic over C. $\diamondsuit$

Theorem 2.4: Let $K \subseteq L$ be differential fields, $C_K = C_L = C$. Let $u_1, \ldots, u_m$, $v_1, \ldots, v_n \in L$, $u_j \neq 0$ for $1 \leq j \leq m$. Suppose that for each $i = 1, \ldots, n$

$$\sum_{j=1}^{m} c_{ij} \frac{u_j'}{u_j} + v_i' \in K\,, \quad c_{ij} \in C \quad \text{for } 1 \leq j \leq m\,.$$

Then either the transcendence degree

$$\text{tr deg}(K(u_1, \ldots, u_m, v_1, \ldots, v_n)/K) \geq n$$

or the n elements

$$\sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j} + dv_i \in \Omega_{L/K}\,, \quad 1 \leq i \leq n$$

are linearly dependent over C. $\diamondsuit$

Notice the similarity of the above theorem with Schanuel's conjecture.

Proof of Theorem 2.2: Let $I = E \cup \Lambda$. Define

$$u_i = \begin{cases} \theta_i & \text{if } i \in E \\ \lambda_j & \text{if } j \in \Lambda \end{cases}, \qquad v_i = \begin{cases} \eta_i & \text{if } i \in E \\ \theta_j & \text{if } j \in \Lambda \end{cases}.$$

Then $u_i'/u_i - v_i' = 0$ for all $i \in I$. We permute the indices such that $I = \{1, \ldots, p\}$. The transcendence degree

$$\text{tr deg}(C(x, u_1, \ldots, u_p, u, v_1, \ldots, v_p, v)/C) = p + 1$$

since $u, v \in K_n$ and exactly one of the u_i, v_i, $i \in I$ is transcendental over K_{i-1}. Hence by theorem 2.4 applied to the following elements from C,

$$x' = 1\,, \quad \frac{u'}{u - v'} = 0\,, \quad \frac{u_i'}{u_i} - v_i' = 0\,, \quad i \in I\,,$$

we conclude that the following elements from $\Omega_{K_n/C}$,

$$dx\,, \quad \frac{1}{u}du - dv\,, \quad \frac{1}{u_i}du_i - dv_i\,, \quad i \in I\,,$$

are linearly dependent over C. In fact, $du/u - dv$ must depend linearly on dx, $du_i/u_i - dv_i$, that is there exist constants γ, $\gamma_1, \ldots, \gamma_p \in C$ such that

$$\frac{du}{u} - dv + \sum_{i=1}^{p} \gamma_i \left(\frac{du_i}{u_i} - dv_i \right) + \gamma dx = 0\,. \tag{$*$}$$

Let $c_0 = 1$, $c_1, \ldots, c_q$ be a vector space basis for the $\mathbb{Q}$-span of $\gamma_0 = 1$, $\gamma_1, \ldots, \gamma_p$ over $\mathbb{Q}$ and write

$$\gamma_i = \sum_{j=0}^{q} n_{ij} c_j\,, \quad 1 \le i \le p\,, \quad n_{ij} \in \mathbb{Q}.$$

We can adjust the c_j be dividing them by the common denominator of n_{ij} such that the new $n_{ij} \in \mathbb{Z}$. We now rewrite $(*)$ to

$$\sum_{j=0}^{q} c_j \left(\frac{d\left(u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}\right)}{u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}} - d(n_{0j} v + n_{1j} v_1 + \ldots + n_{pj} v_p) \right) + \gamma dx = 0\,.$$

For $j = 0, \ldots, q$ let $z_j = u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}$, and $y_j = n_{0j} v + n_{1j} v_1 + \ldots + n_{pj} v_p$. We have $z_j'/z_j = y_j'$ and

$$\sum_{j=0}^{q} c_j \frac{dz_j}{z_j} - d\left(\sum_{j=0}^{q} c_j y_j - \gamma x \right) = 0\,.$$

Since the c_j are linearly independent over $\mathbb{Q}$, by theorem 2.3 we conclude that each z_i and $w = \sum_{j=0}^{q} c_j y_j - \gamma x$ are algebraic over C, hence constants. Thus $y_j' = z_j'/z_j = 0$ and for $j = 0$.

$$n_{00} v = y_0 - n_{10} v_1 - \ldots - n_{p0} v_p\,.$$

Dividing by $n_{00} \neq 0$ leads to the statement. $\diamondsuit$